Sweeping x-monotone pseudolines*

Therese Biedl† Erin Chambers‡ Irina Kostitsyna§ Günter Rote¶

1 Introduction

Consider an arrangement $\mathcal{A}$ of n x-monotone infinite curves where each pair of curves crosses exactly once. These define a directed acyclic planar graph $G_{\mathcal{A}}$, by replacing each crossing with a new vertex, adding two vertices s, t at negative and positive infinity, and directing edges left-to-right. This paper concerns the problem of sweeping the arrangement with a rope of short length, or equivalently, sweeping $G_{\mathcal{A}}$ with a sequence of short st-paths. Formally, we start with a rope at the lower hull of the arrangement. At each step, whenever the rope contains the bottom chain of an inner face F, we may *flip* across F by replacing the bottom chain by the top chain of F. We stop when the rope is the upper hull. The *rope-length* of such a sweep is the maximum length of the rope, measured as the number of edges in the graph. See Figure 1.

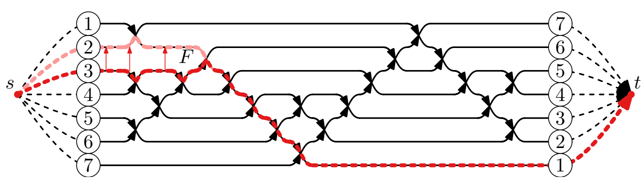

Figure 1: A pseudoline arrangement $\mathcal{A}$ with x-monotone curves and the corresponding graph $G_{\mathcal{A}}$. Rope π (red dashed) has length 8 and can be flipped across face F.

One can easily construct an arrangement $\mathcal{A}$ where the lower hull has length n, so we cannot in general hope to find a sweep of rope-length less than n. But can we always achieve rope-length $n+O(1)$ with a suitable sweep? We show that this is false: for some arrangements we need rope-length at least $\frac{7}{4}n - \frac{5}{4}$. We also provide an asymptotically matching upper bound: For any such arrangement $\mathcal{A}$, we can find a sweep with rope-length at most $2n - 2$. Furthermore, the sweep has special properties: we simultaneously sweep the dual graph $G^*_{\mathcal{A}}$ of $G_{\mathcal{A}}$, and the two ropes of the two sweeps "hug" in some sense.

Finally, we study hardness results. A rope in $G_{\mathcal{A}}$ corresponds to an edge-cut in $G^*_{\mathcal{A}}$, and sweeping with a rope hence corresponds to finding a vertex order that has small cuts. This is the *cutwidth* problem, and since we impose special conditions on the graph and the sweep, our problem is equivalent to solving Directed Cutwidth in $G^*_{\mathcal{A}}$ (definitions and details are in Section 5). Surprisingly enough, we have not been able to find NP-hardness results for this problem, especially not in planar graphs. We therefore show that Directed Cutwidth is NP-hard even in planar graphs with maximum degree 6. Unfortunately the graphs constructed in the reduction are not duals of pseudoline arrangements, so the complexity of minimizing the rope-length in our sweeping problem remains open.

Related results: The problem of minimizing the rope-length of a sweep is motivated by the problem of enumerating all arrangements of n pseudolines [14]. An easy upper bound on the rope-length in a sweep is the maximum length of an x-monotone st-path. However, this does not lead to a good upper bound: x-monotone paths can have close to n^2 edges [2, 11], see [10] for related results. This shows that it is necessary to choose a sweep carefully.

The idea of "sweeping a plane graph" is closely related to the so-called *homotopy height*, see [3, 7, 13] for an overview. Here we are given an undirected planar graph G with a fixed planar embedding and two vertices s, t on the outer-face. We are asked to find a sequence of st-paths that begin and end with the two st-paths that run along the outer-face. Consecutive st-paths in the sequence must be related via a limited set of *permitted operations*, which include flipping across a face and introducing or eliminating a spike along an edge. The goal is to minimize the maximum path-length in the sequence. Our problem is hence the same as computing the homotopy height, except that we restrict the set of permitted operations since the path must follow the edge directions.

Computing the homotopy height of a graph is in NP

*Research initiated during the 2020 Workshop on Graphs and Geometry at the Bellairs Research Institute. The authors would like to thank all participants of the workshop, but especially Hugo Akitaya and Stefan Felsner, for helpful input.

†David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada. Supported by NSERC.

‡Department of Computer Science and Engineering, University of Notre Dame, Indiana, USA. Research supported in part by NSF awards 1907612 and 2444309.

§KBR at NASA Ames Research Center, Moffett Field, CA, USA.

¶Freie Universität Berlin, Institut für Informatik, Germany.

[7], but it remains open whether this problem is NP-hard. There is also a relationship between the homotopy height and the height of a planar straight-line grid-drawing [3]; in particular every N-vertex planar graph G has homotopy height at most $\frac{2}{3}N + O(1)$ since G has a planar straight-line grid-drawing where the smaller dimension is $\frac{2}{3}N + O(1)$ [8]. Unfortunately, this does not help to solve our problem, for two reasons. First, in our sweeps we impose stronger restrictions on when we are allowed to flip across a face. Second, we are sweeping an arrangement of n curves, hence the corresponding planar graph has $N \in \Theta(n^2)$ vertices and the above bounds are meaninglessly big.

As mentioned earlier, sweeping a pseudoline arrangement $\mathcal{A}$ with a short rope corresponds to solving Directed Cutwidth in the dual graph $G^*_{\mathcal{A}}$. The (undirected) version Cutwidth of this problem is very well-established in the literature and is known to be NP-hard even in planar graphs with maximum degree 3 [12]. Cutwidth is also SSE-hard to approximate within any constant factor [15]. SSE stands for the *Small Set Expansion conjecture*; we refer to this paper for the definition of "SSE-hard" and other results concerning cutwidth.

2 Definitions

Throughout the paper, $\mathcal{A}$ denotes a set of n x-monotone infinite curves that form a *pseudoline arrangement*, i.e., each pair of curves has exactly one point in common where the curves properly cross. The curves in $\mathcal{A}$ are called *pseudolines*. Arrangement $\mathcal{A}$ naturally defines a planar graph $G_{\mathcal{A}}$, by replacing every crossing between pseudolines by a vertex, adding an edge whenever two crossings are consecutive on a pseudo-line, adding two vertices s and t that represent the points at negative and positive infinity, and connecting s to the first crossing and t to the last crossing of each pseudo-line. We direct all edges of $G_{\mathcal{A}}$ from left to right, making it a directed acyclic planar graph with exactly one source s and one sink t that are both on the outer-face. Such a graph is known as a *bipolar orientation*, and many properties are known, see for example [9]. In particular, for any inner face F, the boundary consists of two directed paths; in our situation where edges are drawn left-to-right these paths naturally are called the *top chain* and *bottom chain* of F. Their common start-vertex is the *source* $s(F)$ of F, and their common end-vertex is the *sink* $t(F)$ of F. At any vertex $v \neq s, t$, the incoming edges are consecutive in the clockwise order around v, as are the outgoing edges. In our situation with edges drawn left-to-right, we can naturally speak of the topmost/bottommost incoming/outgoing edge of a vertex.

A *rope* of $\mathcal{A}$ is a directed st-path π in $G_{\mathcal{A}}$; alternatively we can view π as an x-monotone infinite curve along pseudo-lines. For any two points p, p' on π, we use $\pi(p, p')$ to denote the sub-curve between the two points (including p, p'). If π contains the entire bottom chain of some inner face F, then *flipping rope* π *across* F means to create a new rope that is π except that the bottom chain $\pi(s(F), t(F))$ of F gets replaced by the top chain of F. A *sweep* of $\mathcal{A}$ consists of a sequence $\pi_1, \ldots, \pi_k$ of ropes where π_1 is the lower hull of $\mathcal{A}$, π_k is the upper hull of $\mathcal{A}$, and consecutive ropes are obtained by flipping across an inner face. The *rope-length* of such a sweep is the maximum length (measured by the number of edges) among the used ropes, and the problem studied in this paper is to find a sweep that has small rope-length.

Graph $G_{\mathcal{A}}$ (and generally any bipolar orientation) naturally gives rise to a dual graph $G^*_{\mathcal{A}}$ that is also a bipolar orientation as follows. Temporarily add an edge (s, t) to $G_{\mathcal{A}}$, and let s^*, t^* be the two faces incident to it, with s^* incident to the upper hull of $\mathcal{A}$. The vertices of $G^*_{\mathcal{A}}$ are now s^*, t^*, and one vertex F for each inner face of $G_{\mathcal{A}}$. For every edge $e = u \to v$ of $G_{\mathcal{A}}$, let F_ℓ and F_r be the faces that lie to the left and right when walking from u to v. (Since our edges are directed left-to-right, these faces are really above and below e, but "left"/"right" is the established term in the literature.) We add to $G^*_{\mathcal{A}}$ the *dual edge* e^* *of* e, which is $F_\ell \to F_r$. Note that e lies on the top chain of F_r and the bottom chain of F_ℓ, so in any sweep we must have swept F_r *before* we can sweep F_ℓ. We think of dual graph $G^*_{\mathcal{A}}$ as drawn such that each vertex F is placed in the corresponding face of $G_{\mathcal{A}}$, and each edge e^* crosses the edge e that it is dual to. By definition, e^* crosses e from left to right.

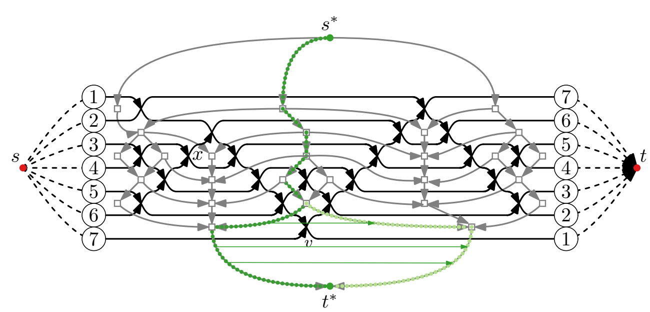

Figure 2: The dual graph $G^*_{\mathcal{A}}$ with a dual rope π^* (green dotted) that can be flipped across vertex v.

Since $G^*_{\mathcal{A}}$ is also a bipolar orientation, concepts such as "rope" and "flipping across a face" can also be applied to $G^*_{\mathcal{A}}$. For ease of distinction, we use the term *dual rope* for a rope in $G^*_{\mathcal{A}}$, and *flipping across a vertex (of* $G_{\mathcal{A}}$*)* for the operation of flipping across a face of $G^*_{\mathcal{A}}$. Note that any dual rope π^* defines an st-cut by virtue of taking the edges of $G_{\mathcal{A}}$ that it *crossed* (i.e., whose duals it contained), and symmetrically every rope

π defines an s^*t^*-cut. Both these cuts are *directed*, i.e., contain only edges directed from the source-side to the sink-side.

3 A lower bound

Theorem 1 *For $n = 3 \bmod 4$, there exists a pseudoline arrangement $\mathcal{A}$ of n x-monotone curves such that any sweep requires rope-length at least $\frac{7}{4}n - \frac{5}{4}$.*

Proof. The construction is symmetric, and we describe it from left to right, see Figure 3 for the construction for $n = 7$ and Figure 7 (in the appendix) for $n = 15$. Start with two curves c, c' (black solid) that are at the top and bottom at the far left and intersect in some point x. All other curves will pass above x. Set $K = \frac{n-3}{4}$. Between c and c' at the far left are $2K + 1$ "top" curves (red, dashed) at even positions, and $2K$ "bottom" curves (blue, dotted) at odd positions.

In the beginning, the red curves move up and the blue curves move down until they are separated, forming a $2K \times 2K$ half-grid (shown shaded in Figure 3). So far there are no intersections between curves of the same color. In the area below all red curves and above all blue curves, there are three faces F_ℓ, F_c, and F_r, separated from each other by c and c'.

Before the $2K{+}1$ red curves cross c, we let the lower $K{+}1$ of them cross each other in such a way that they all become incident to the top chain of F_ℓ. These curves, together with c, hence create a $(K{+}1) \times (K{+}1)$ half-grid. (In terms of sorting networks, this half-grid is the *bubble-sort* network.)

In the middle, in the area above x, we do two things: a) We cross the blue curves in such a way that they all become incident to the bottom chain of F_c, forming a $(2K{+}1) \times (2K{+}1)$ half-grid together with c and c'. b) We cross the upper K with the lower K red curves (the middle red curve remains uncrossed, as it meets all other red curves in the half-grids above F_ℓ and F_r).

The right part of the construction is symmetric. As shown in Figure 8 in the appendix, this arrangement can even be drawn with straight lines. Observe the following properties of x-monotone paths in the construction:

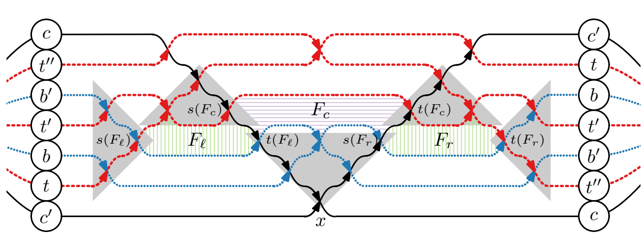

Figure 3: Construction for the lower bound for $n = 7$ (so $K = 1$); we need rope-length 11.

- Any x-monotone path from s to the source $s(F_\ell)$ of F_ℓ has length at least $2K$. This holds because such a path must traverse the $2K \times 2K$ half-grid, plus the edge from s to reach the half-grid.
- Any x-monotone path π from $t(F_c)$ to t has length at least $2K + 1$. This is obvious if π walks along c' until the intersection with the last red curve (and from there to t). So assume that it walks along c' for $i < 2K$ edges and then turns onto a red curve that brings us (perhaps after some more edges) to the half-grid right of $t(F_r)$. It then traverses a $(2K{-}i) \times (2K{-}i)$ half-grid, which takes $2K{-}i$ edges, plus one more edge to t. Hence the path has length at least $2K{+}1$.
- Any x-monotone path from $t(F_\ell)$ to $s(F_r)$ has length at least $2K$, because it must go across the $(2K{+}1) \times (2K{+}1)$ half-grid below F_c and can (at best) use shortcuts along the bottom chain of F_c.

Now we come to the actual proof. Consider any sweep of $\mathcal{A}$. Since the dual graph has edges $F_c \to F_\ell$ and $F_c \to F_r$, we must flip across both F_ℓ and F_r before flipping across F_c. By symmetry we may assume that we flip across F_r first, and consider the rope π immediately after we flipped across F_ℓ. Then π goes from s to $s(F_\ell)$, from there along the top chain of F_ℓ to $t(F_\ell)$, from there to $s(F_r)$ and $t(F_c)$ (since we have flipped across F_r but not F_c yet), and from there to t. So

$$\begin{aligned} |\pi| &= |\pi(s, s(F_\ell))| + \text{length of top chain of } F_\ell \\ &\quad + |\pi(t(F_\ell), s(F_r))| + 1 + |\pi(t(F_c), t)| \\ &\geq 2K + K{+}2 + 2K + 1 + 2K{+}1 = 7K + 4 \end{aligned}$$

which is at least $7\frac{n-3}{4} + 4 = \frac{7}{4}n - \frac{5}{4}$. □

4 An upper bound: The primal-dual sweep

We now show an upper bound on the required rope-length by defining a sequence of ropes in $G_{\mathcal{A}}$ and simultaneously a sequence of dual ropes that "hug" the ropes. To define this, we first need a few other definitions and observations about a rope π and a dual rope π^* (see also Figure 4).

Rope π connects s to t, hence must go across the directed st-cut defined by π^*, and can do so only once since π is directed. It follows that exactly one edge e of π is crossed by π^*; we call e the *active edge* and let x be the point where it is crossed by π^*. This *crossing-point* x splits the rope into two parts $\pi(s, x)$ and $\pi(x, t)$, and likewise splits the dual rope into $\pi^*(s^*, x)$ and $\pi^*(x, t^*)$, and the properties that we require will depend on which part we are in.

Definition 1 *We say that a rope π and dual rope π^** hug *each other if the following four (symmetric) conditions hold: (1) for every edge e in $\pi(s, x)$, the face*

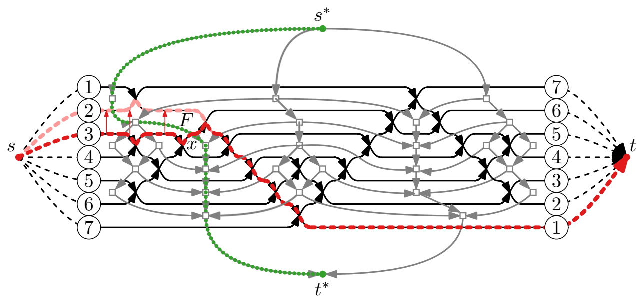

Figure 4: A rope and a dual rope that hug each other. We can flip across face F, which is to the left of the active edge.

to the left of e belongs to π^*; (2) for every edge e in $\pi(x,t)$, the face to the right of e belongs to π^*; (3) for every edge e^* in $\pi^*(s^*,x)$, the face of $G^*_{\mathcal{A}}$ (hence vertex of $G_{\mathcal{A}}$) to the right of e^* belongs to π; (4) for every edge e^* in $\pi^*(x,s^*)$, the vertex of $G_{\mathcal{A}}$ to the left of e^* belongs to π.

We will now define a sequence of *rope pairs* (i.e., pairs of a rope π and a dual rope π^*) such that the ropes sweep $G_{\mathcal{A}}$, the dual ropes sweep $G^*_{\mathcal{A}}$, and at all times π and π^* hug each other. Then we argue that this implies rope-length at most $2n-2$ at all times. We initialize rope π as the lower hull of $\mathcal{A}$, so all edges of π have t^* to their right. We initialize the dual rope π^* to contain all faces incident to s, in order from top to bottom, so all edges of π^* have s to their right. The active edge is the bottommost outgoing edge of s, and one easily verifies all conditions. (The appendix shows an example of a sweep from the beginning.) To explain how to update the rope pair, we need some observations.

Claim 1 *(1) At any vertex $v \neq s$ of $\pi(s,x)$, rope π uses the top incoming edge. (2) At any vertex $v \neq t$ of $\pi(x,t)$, rope π uses the bottom outgoing edge. (3) At any face $F \neq s^*$ of $\pi^*(s^*,x)$, dual rope π^* crosses the first edge of the top chain of F. (4) At any face $F \neq t^*$ of $\pi^*(x,t^*)$, dual rope π^* crosses the last edge of the bottom chain of F.*

Proof. We only prove the first claim, the other three are symmetric. Let e be the incoming edge of v on π, and assume for contradiction that e is not top incoming. Then the face F to the left of e is incident to two incoming edges of v, hence $v = t(F)$ and e is the last edge of the bottom chain of F. By the hugging-condition F belongs to π^*; the next edge on π^* hence crosses the bottom chain of F. But then $v = t(F)$ is on the t-side of the st-cut defined by the dual rope π^*, contradicting that $v \in \pi(s,x)$. □

Claim 2 *Let e be the active edge and let v be its head and F be the face to its left. If $F \neq s^*$ or $v \neq t$, then we can flip π across F or flip π^* across v, and the new pair of rope and dual rope hug each other.*

Proof. The claim is illustrated in Figure 5. Assume first that e is not top incoming, which implies that it is the last edge of the bottom chain of F. We know that $F \neq s^*$ since all edges incident to s^* are top incoming. Since $\pi(s,x)$ only uses top incoming edges, π must have traversed the entire bottom chain of F and by $F \neq s^*$ we can hence flip across F to get the new rope π'. The new active edge is the first edge of the top chain of F by Claim 1(3). The hugging-conditions could be violated only at face F (everywhere else the rope and dual rope are unchanged), and one easily verifies that they hold here because all new edges of π' have F to their right.

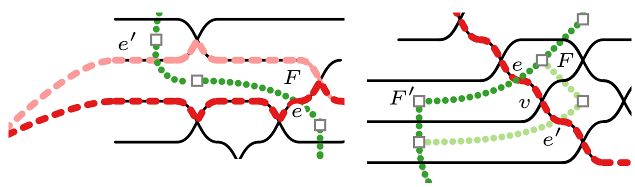

Figure 5: Closeup of flipping across a face and a vertex. Dual graph not shown.

Now assume that e is top incoming, which implies that $v \neq t$ since otherwise $F = s^*$ and not both are allowed. Let F' be the face to the right of e; this is in $\pi^*(x,t)$ since e (as active edge) is crossed by π^*. All other incoming edges of v are the last edge of the bottom chain of the faces to their left. Applying Claim 1(4) repeatedly, starting with $F' \in \pi^*(x,t^*)$, therefore dual rope π^* must cross all incoming edges of v. So by $v \neq t$ we can flip the dual rope across v. By Claim 1(2) rope π continues from v along the bottom outgoing edge, which hence becomes the new active edge. Again one easily verifies the hugging condition, since all new edges of the new dual rope have v to their right. □

We hence update π and π^* as follows. Let e be the active edge, and let v be its head and F be the face to its left. If $F = s^*$ and $v = t$ then e is the last edge of the upper hull. By Claim 1(1) hence π is the upper hull and the sweep is finished. By Claim 1(4) π^* crosses all incoming edges of t, and so the sweep of the dual is also finished. Otherwise (either $F \neq s^*$ or $v \neq t$) we perform one of the flips that exists by Claim 2 and repeat.

4.1 Analysis

The sweeping algorithm as described would actually work for any bipolar orientation. We now show that if the bipolar orientation comes from a pseudoline arrangement $\mathcal{A}$ of x-monotone curves, then the rope-length is at most $2n-2$ at all times. Enumerate the pseudolines from top to bottom in the order of incidence with s as $c_1, \ldots, c_n$. The *index* of an edge e is the index of the pseudoline that supports e, i.e., along which e runs. The following observation is trivial (it holds since pseudolines intersect only once, so one can go above the other only once), but will be crucial for counting vertices later.

Observation 1 *At any vertex $v \neq s, t$, the indices of incoming edges increase from top to bottom, while the indices of outgoing edges decrease from top to bottom.*

An *encounter* of rope π with pseudoline c_i is a maximal sub-curve $\pi(v, v')$ that belongs to c_i. Note that v, v' are necessarily vertices, and possibly $v = v'$.

Corollary 1 *While walking along $\pi(s, x)$, the index i of the current edge of π can only increase, and any pseudoline c_j encountered at the next vertex v satisfies $j \geq i$.*

Proof. Rope π enters along the top incoming edge of v, hence i is the smallest index of a pseudoline incident to v. So all pseudolines encountered at v (including the one along which π leaves) cannot have smaller index. □

Claim 3 *While walking along $\pi(s, x)$, we encounter every pseudo-line at most once.*

Proof. Assume for contradiction that we encounter pseudoline c_i at least twice. At the end of the first encounter we hence have a vertex v with $v \in c_i \cap \pi(s, x)$, but π continues beyond v along some pseudoline c_j with $j \neq i$. If $j > i$, then the index throughout $\pi(v, x)$ is at least $j > i$, and so we cannot encounter c_i again. So we must have $j < i$, which means that the outgoing edge of π at v is *below* the outgoing edge along c_i by Observation 1. Therefore c_i has entered the s^*-side of the s^*t^*-cut defined by π. Since $\pi(s, x)$ always uses top incoming edges, there are no edges from the s^*-side to $\pi(s, x)$, and so c_i cannot encounter $\pi(s, x)$ again. □

Claim 4 *At any time during the sweep, rope π has length at most $2n - 2$.*

Proof. Assign to s the pseudoline along which π leaves, and assign to every vertex $v \neq s$ on $\pi(s, x)$ the pseudoline c that supports the bottom incoming edge e at v. This assigns every pseudoline at most once, for e was *not* in $\pi(s, x)$ by Claim 1, and so v is the beginning of the unique encounter of c with $\pi(s, x)$. (This also shows that c was not assigned to s). So $\pi(s, x)$ has at most n vertices, and symmetrically $\pi(s, t)$ has at most n vertices and the rope-length is at most $2n - 1$.

We claim that this is not tight. Assume for contradiction that at some point rope π has length exactly $2n-1$, so $\pi(s, x)$ has n vertices and *all* pseudolines have been assigned to some vertex of $\pi(s, x)$. Observe that c_1 must have been assigned to s, for otherwise the index of $\pi(s, x)$ would be greater than 1 throughout, so $\pi(s, x)$ could not encounter c_1, so c_1 would not be assigned to a vertex. Also observe that c_2 must have been assigned to a vertex v that lies on c_1, because it is not assigned to s, and we assign (by Observation 1 and Claim 1(1)) a pseudoline c_j to a vertex $v \neq s$ only if $\pi(s, x)$ has index less than j when it reaches v. In particular therefore c_1 and c_2 intersect at a point on $\pi(s, x)$. By a completely symmetric argument, c_1 and c_2 intersect again at a point on $\pi(x, t)$. This is not possible in a pseudoline arrangement. □

Theorem 2 *For every pseudoline arrangement of n x-monotone curves, there exists a sweep with rope-length at most $2n - 2$.*

A few comments are in order. First, as the example shown in the appendix illustrates, the bound is tight: for some arrangements this particular method of computing a sweep requires rope-length $2n - 2$.

Also, our coordinated primal-dual sweep can be interpreted as a *left-first greedy* sweep: At each stage, the rope π selects the leftmost possible position where it can flip over a face. The dual rope π^* can be interpreted as guiding the search for the sweep position: As long as a flip is not possible at the current position of the active edge, the active edge advances to the right, and this corresponds to a dual flip. Such a left-first greedy method was used in an algorithm by Alvarez and Seidel as a tool to count the number of triangulations [1].

5 NP-hardness

In this section, we reduce our sweep-problem to solving Directed Cutwidth in $G^*_{\mathcal{A}}$. Then we show that Directed Cutwidth is NP-hard even in planar graphs with maximum degree 6. Unfortunately this does not prove the sweep-problem NP-hard since the graph that we construct cannot be the dual graph of a pseudoline arrangement (it has vertices of degree 2 and many sources and sinks).

We need a few definitions. Fix a vertex order $\sigma = \langle v_1, \ldots, v_n \rangle$ of G. For $1 \leq i \leq n$, the *ith cut* (or *cut after v_i*) is the set of edges (v_h, v_j) with $h \leq i < j$. The maximum cardinality of these cuts is the *width* of the vertex order, and the *cutwidth* of graph G is the minimum width over all vertex orders.

The cutwidth is defined for undirected graphs, but for directed acyclic graphs there exists a natural restriction, apparently first studied in [4]: The *directed cutwidth* of a directed acyclic graph G is the minimum width of a vertex order of G that is a *topological order*, i.e., where every edge is directed from a lower-indexed to a larger-indexed vertex.

Lemma 3 *Let $\mathcal{A}$ be a pseudo-line arrangement with x-monotone curves. Then $\mathcal{A}$ has a sweep with rope-length at most w if and only if $G^*_{\mathcal{A}}$ has directed cutwidth at most w.*

Proof. We only show one direction, the other is similar. Fix a sweep with rope-length w. This defines a sequence $\sigma = \langle F_1, \dots, F_k \rangle$ of the inner faces of $G_{\mathcal{A}}$ via the order in which the sweep flips the rope across faces. We append $s^* =: F_{k+1}$ and pre-pend $t^* := F_0$ to this sequence since the rope begins incident to t^* and ends incident to s^*. Sequence σ hence gives a vertex order $F_0, F_1, \dots, F_{k+1}$ of $G^*_{\mathcal{A}}$. Any directed edge $F_\ell \to F_r$ of $G^*_{\mathcal{A}}$ is dual to an edge e of $G_{\mathcal{A}}$ that is on the upper chain of F_r and the lower chain of F_ℓ. So the sweep must flip across F_r *before* flipping across F_ℓ, i.e., $r < \ell$. So in our face order all edges of $G^*_{\mathcal{A}}$ are directed right-to-left, and reversing it (which does not affect the width) gives a topological order. Finally the edges of the ith cut are dual to the edges of the rope after flipping across F_i, and vice versa. Therefore the width of the topological order is the same as the rope-length. □

So we are interested in the complexity of problem Directed Cutwidth, the decision version of the problem: Given a directed acyclic graph G and an integer w, is there a topological order of width at most w? Surprisingly, the complexity of this problem does not appear to have been studied much in the literature. Wu et al. [15] showed that Directed Cutwidth (not specifically named there, but appearing in row 6 of their Table 1) is SSE-hard to approximate (the constructed graphs are non-planar). There are also some positive results; in particular Directed Cutwidth has a linear-time algorithm if w is a constant [4], and for series-parallel graphs it can be computed in quadratic time [5]. But we have the following new result:

Theorem 4 Directed Cutwidth *is NP-hard, even in planar graphs with maximum degree 6.*

Proof. The reduction is from Cutwidth, which is known to be NP-hard, even for a planar graph with maximum degree 3 [12]. So assume that we are given a planar graph G with maximum degree 3 and an integer w and we want to test whether its cutwidth is at most w. We may assume that G has no isolated vertices or isolated edges: They do not affect the cutwidth, except in the trivial case that G consists exclusively of isolated edges and vertices. We create a directed graph H as follows (see Figure 6). We retain all vertices of G, and replace every edge $e = (v, w)$ by a source s_e and a sink t_e that are both incident to both v, w. (A similar transformation, using only a sink, was used in [15].)

We claim that G has a vertex order σ_G of width at most w if and only if H has a topological order σ_H of width at most $2w + 2$. We sketch here a proof; details

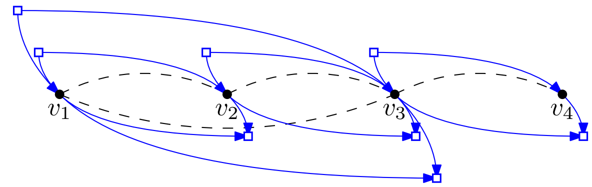

Figure 6: From a vertex order of G (black dashed) to a topological order of H (blue solid). For ease of reading we offset sources to be above and sinks to be below vertices of G.

are in the appendix. To convert σ_G to σ_H, simply add (for each edge e of G) source s_e just before the first endpoint of e in σ_G, and sink t_e just after the second endpoint of e in σ_G. Elementary arguments (using that G has maximum degree 3) show that σ_H then has width at most $2w + 2$. To convert σ_H to σ_G, initially simply take the induced vertex order, which is easily seen to have width at most $w + 1$. This can be tight (say at the ith cut) only if v_i has no neighbours on the left while v_{i+1} has no neighbours on the right. Call such a pair (v_i, v_{i+1}) *improvable*: exchanging the two vertices in the order improves the size of the ith cut and leaves all other cuts after vertices unchanged. Exchanging all improvable pairs hence gives the desired σ_G. □

6 Summary and outlook

In this paper, we studied the problem of sweeping a pseudoline arrangement of n x-monotone curves using a rope between the points of infinity. The only permitted move is to flip parts of the rope from the bottom chain to the top chain of a face, and the goal is to keep the number of edges on the rope small. We argue that the worst-case rope-length is in $\Theta(n)$, and specifically, at most $2n - 2$ (for all arrangements) and at least $\frac{7}{4}n - \frac{5}{4}$ (for some arrangements).

The most tantalizing open problem is the complexity of finding the shortest rope, possibly for an arbitrary bipolar orientation instead of a pseudoline arrangement. We proved NP-hardness of Directed-Cutwidth, which is closely related to our problem via duality. But the graph that we construct for the NP-hardness ihas many sources and sinks, and so is not the dual graph of a pseudoline arrangement, and proving NP-hardness of the original problem or finding a polynomial-time algorithm for it remains open.

Our sweep by definition is monotone in the sense that every inner face is swept exactly once. Could a shorter rope-length ever be achieved if we are permitted to reverse some flips? We suspect that (as for the homotopy height under some restrictions on the input [6]) repeatedly sweeping a face cannot shorten the rope-length, but this remains open.

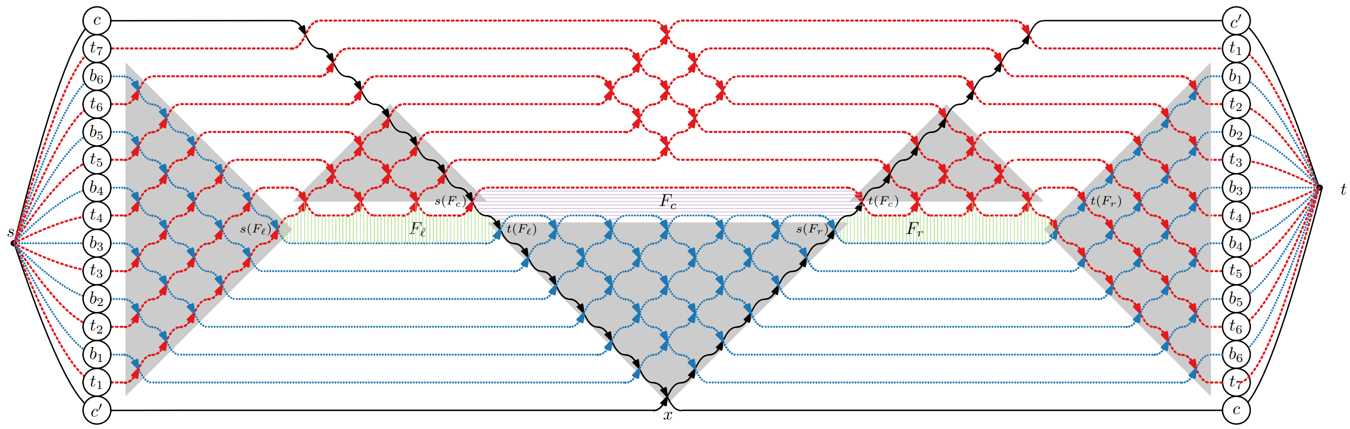

Figure 7: The lower-bound construction for $n = 15$ pseudolines ($K = 3$); we need rope-length 25.

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A More details on lower bounds

Figure 7 shows the lower bound example for $n = 15$. Figure 8 shows that it can be realized even as a line arrangement.

The greedy algorithm will actually achieve ropelength $(7n - 5)/4 = 7K + 4$ in these instances.

B More details on "Upper bounds"

We illustrate another example of how the sweep is performed in the following sequence of figures. The construction consists of n pseudolines $c_1, \dots, c_n$, enumerated in top-to-bottom order at s, that satisfy the following:

- For any $i > 1$, the first crossing along c_i is with pseudoline c_1.
- Let F be the face to the left of the last edge of c_1. Then the top chain of F meets all pseudolines except c_1.

See Figure 9 for the pseudoline arrangement (for $n = 7$) and the initial rope and dual rope.

We show that if these conditions hold, then the rope-length becomes $2n - 2$ at some point (hence the bound of Claim 4 is tight). To see this, observe that the first move is to flip across a face, since the active edge (which is the first edge of pseudoline c_n) is bottom incoming. See Figure 10.

The next few moves will *all* be face-flips, because the active edge is always the first edge of pseudoline c_i for some $i > 1$, which is bottom incoming because it ends at the intersection with c_1. So we continue face-flips until the active edge is the first edge of c_1, and in fact the entire rope is exactly c_1. See Figure 11.

Now the active edge is on c_1, hence top incoming, and we do a vertex-flip, which pushes the active edge one further down the rope (i.e., along c_1). See Figure 12.

The next few moves will actually *all* be vertex-flips, because the active edge is always on c_1, hence top-incoming if its head is not t. So we continue doing vertex-flips until the active edge is the last edge of c_1. See Figure 13.

Now the active edge is bottommost incoming at its head t, which means that we do a face-flip at the face F to the left of the active edge. Recall that we constructed our arrangement so that the upper chain of this face F has length $n - 1$. Also, pseudoline c_1 has n edges, of which the rope uses all but the last one. Therefore at this point the rope has length $2n - 2$. See Figure 14.

We note that rope-length $2n-2$ is not required in this example if we sweep differently. In particular, a sweep with rope-length $n + 1$ can be obtained in this example by applying the algorithm to the reflected arrangement in which left and right are swapped.

C More details on "NP-hardness"

In this section, we fill in the details of the NP-hardness proof of Section 5. Recall given a graph G, we created the directed acyclic graph H by replacing every edge e of G by a source s_e and a sink t_e that both are adjacent to both endpoints of e. This doubles the degrees of all vertices of G (so the maximum degree of H is 6). Also the undirected version of H can be obtained by duplicating all edges of G and then subdividing all edges; in particular if G is planar then so is H.

To argue the bounds on the widths, we need some notation. For any vertex order $v_1, \dots, v_n$ of G, and any $i = 1, \dots, n$, write L_i $[R_i]$ for the set of edges in G that are incident to v_i and whose other endpoint is left [right] of v_i in the vertex order. Also, let B_i be the set of edges that *bypass* v_i, i.e., have the form (v_h, v_j) for $h < i < j$, and note that the cut before and after v_i have size $|B_i| + |L_i|$ and $|B_i| + |R_i|$, respectively.

For both G and H, we write $C^{\leftarrow}(v)$ and $C^{\rightarrow}(v)$ for the cuts directly before and after a vertex v, respectively, and indicate with a subscript which graph this applies to. (The vertex order will be clear from context.)

Claim 5 *If G has a vertex order $v_1, \dots, v_n$ of width w, then H has a topological order σ_H of width at most $2w + 2$.*

Proof. As sketched earlier, σ_H is obtained by inserting, for each edge e, the source just before the left end of e and the sink just after the right end of e. Put differently, for $i = 1, \dots, n$, list all sources of edges of R_i (in arbitrary order), then list v_i, the list all sinks of edges in L_i and proceed to the next i. See Figure 6 for an example, and verify that we indeed obtain a topological order. Also notice that scanning σ_H from left to right, the cut-sizes increase when we pass a source and decrease when we pass a sink, so the maximize cut-size of σ_H must occur immediately before or after some original vertex v_i of G.

One verifies that $C^{\leftarrow}_H(v_i)$ contains exactly two edges each for each edge in $L_i \cup B_i \cup R_i$, due to edges in $C^{\leftarrow}_G(v_i)$ and sources for edges in R_i, respectively. Therefore $|C^{\leftarrow}_H(v_i)| = 2(|L_i| + |B_i| + |R_i|)$, and by symmetry, this is also equal to $|C^{\rightarrow}_H(v_i)|$. Since

$$|B_i| + \max\{|L_i|, |R_i|\} = \max\{|C^{\leftarrow}_G(v_i)|, |C^{\rightarrow}_G(v_i)|\} \leq w,$$

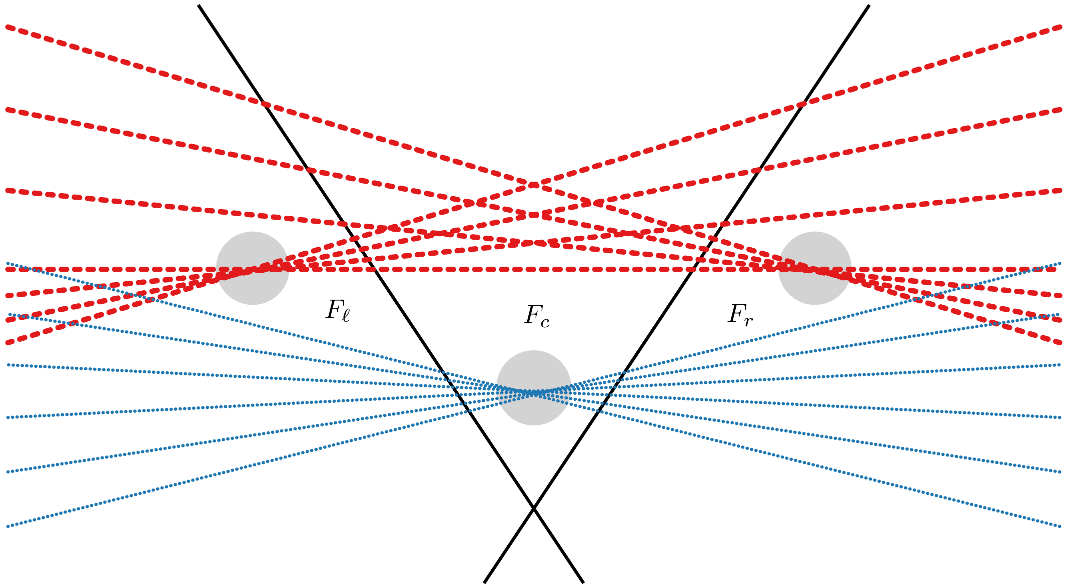

Figure 8: The lower-bound example as an arrangement of straight lines. The slopes of the seven red (dashed) and six blue (dotted) lines are evenly spaced, with red and blue slopes interleaving. This ensures the appropriate intersection pattern when the lines are extended far enough to the left and right. In the three shaded disks, the lines are slightly perturbed from a common intersection point so that they become incident to F_ℓ, F_c, and F_r, respectively.

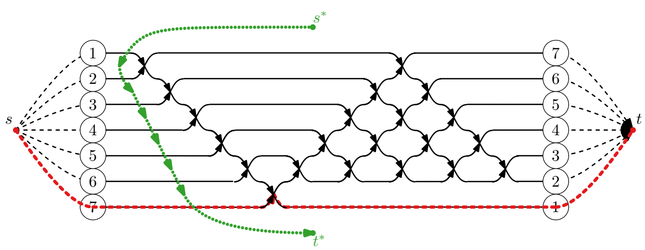

Figure 9: The arrangement, with initial rope and dual rope.

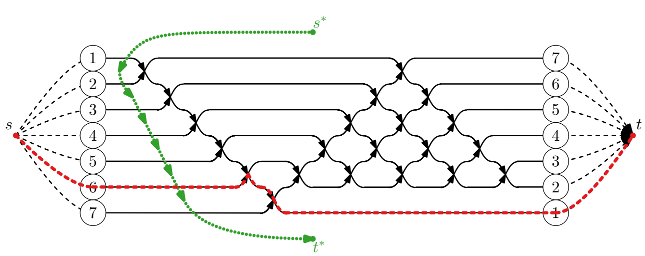

Figure 10: The situation after the first face-flip.

the width of σ_H is at most $2(|B_i| + \max\{|L_i|, |R_i|\} + \min\{|L_i|, |R_i|\}) \leq 2w + 2\min\{|L_i|, |R_i|\} \leq 2w + 2$ since $|L_i| + |R_i| \leq \deg_G(v_i) \leq 3$. □

For the other direction, we must convert a topological order of H into a vertex order of G of small width.

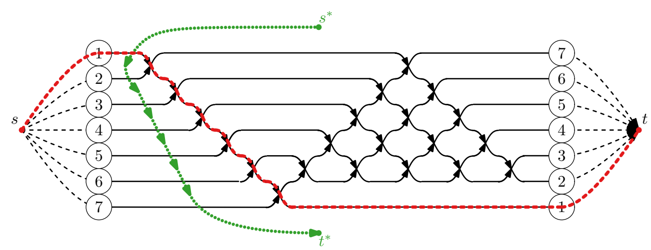

Figure 11: The situation after repeated face-flips until the rope follows c_1.

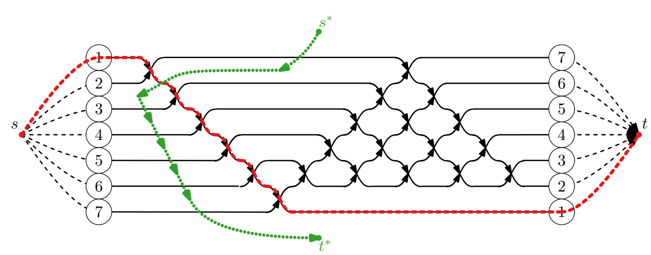

Figure 12: The situation after the first vertex-flip.

Recall that in a vertex order of G, the pair (v_i, v_{i+1}) (for some $1 \leq i < n$) is called an *improvable pair* if $L_i = \emptyset = R_{i+1}$, see also Figure 15.

Claim 6 *If H has a topological order σ_H of width $2w+2$, then in the induced vertex order $v_1, \ldots, v_n$ of G, the*

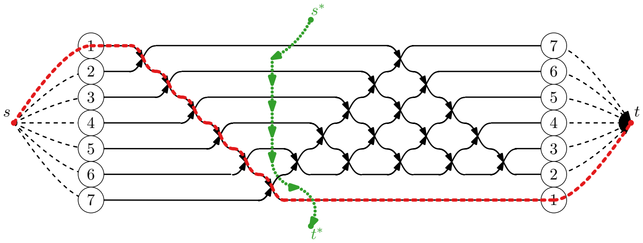

Figure 13: The situation after repeated vertex-flips until the active edge is the last edge of c_1.

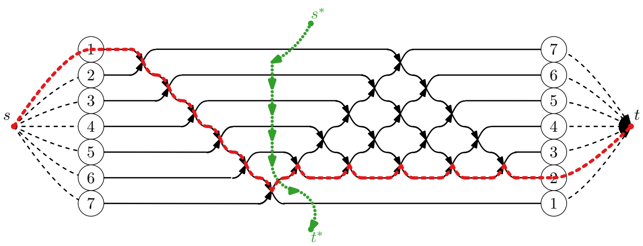

Figure 14: After one more face-flip, the rope length is $2n-2$.

ith cut has width at most $w+1$ *for all* $i<n$, *and equality holds only if* (v_i, v_{i+1}) *is an improvable pair.*

Proof. We have to bound $|B_i|+|R_i|$, and will show that all these edges, *and* the edges of L_i, had contributed to $C_{\vec{H}}(v_i)$, so there cannot be too many of them. For $e \in L_i \cup B_i \cup R_i$, the left end was v_i or farther left, while the right end was v_i or farther right. Since σ_H is a topological order, source s_e was strictly before v_i and sink t_e was strictly after v_i in σ_H, and so in σ_H this contributed two edges to $C_{\vec{H}}(v_i)$. Therefore

$$2w+2 \geq |C_{\vec{H}}(v_i)| \geq 2|L_i| + 2|B_i| + 2|R_i|,$$

which implies that $|C_{\vec{G}}(v_i)| = |B_i| + |R_i| \leq w+1$ and equality can hold only if $L_i = \emptyset$. Symmetrically arguing via the cut before v_{i+1} in σ_H, one sees that

$$2w+2 \geq 2|C_{\overleftarrow{H}}(v_{i+1})| \geq 2(|L_{i+1}| + |B_{i+1}| + |R_{i+1}|)$$

and so $|C_{\vec{G}}(v_i)| = |C_{\overleftarrow{G}}(v_{i+1})| = |L_{i+1}| + |B_{i+1}| \leq w+1$ and equality can only hold if also $R_{i+1} = \emptyset$. □

Figure 15 shows an example where the width of the induced vertex order σ_G is indeed $w+1$. So we are not done yet with the reverse direction of the reduction. But observe that if the pair (v_i, v_{i+1}) is improvable, then by exchanging their order all edges in R_i and L_{i+1} are removed from the cut between them, except the edge v_iv_{i+1} if it exists. Since we have excluded the cases that v_i or v_{i+1} are isolated vertices or v_iv_{i+1} is an isolated edge, the cut strictly improves. All other cuts remain unchanged. We repeat this until no improvable pair remains. In the end, all cut-sizes are at most w as desired, and G has cutwidth at most w.

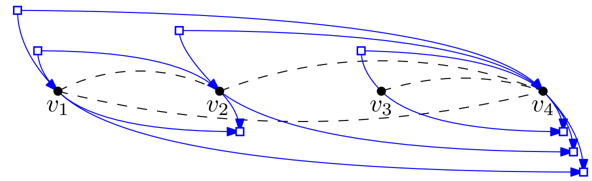

Figure 15: From a topological order of H (blue solid) of width $2w+2=6$ to a vertex order of G (dashed black), but it may not have optimal width: G has cutwidth $w=2$ (see Figure 6), but the cut between v_3 and v_4 has width 3. Note that $L_3 = \emptyset = R_4$, i.e., (v_3, v_4) is improvable.

D Experimental results

We ran some computer experiments, exhaustively trying all pseudoline arrangements with up to $n=9$ pseudolines. Each arrangement was subjected to a rather brute-force attack to find the shortest rope-length, by essentially looking for a path in the graph whose nodes represent all possible ropes. The data that we found are displayed in Table 1. For n of the form $n=4k+3$, the results on the maximum agree with the lower bound of Theorem 1.

n	min	max	#PSLA
2	2	2	1
3	4	4	2
4	5	5	8
5	6	7	62
6	7	9	908
7	8	11	24,698
8	9	12	1,232,944
9	10	14	112,018,190

Table 1: min/max: The shortest and longest rope-length required for pseudoline arrangements with n pseudolines. #PSLA: the number of combinatorial types of x-monotone pseudoline arrangements with n pseudolines (sequence A006245 in the Online Encyclopedia of Integer Sequences).

The lower bound is apparently $n+1$, except for $n=2$. The number of arrangements that require the maximum rope-length grows very quickly. For example, among the arrangements of 7 pseudolines, there are exactly two that require rope-length 11, up to symmetries. On the other hand, with 8 pseudolines, 1184 arrangements among the 1,232,944 arrangements need rope-length 12.