# Straight-line Orthogonal Drawing of Complete Ternary Tree Requires $O(n^{1.032})$ Area

Hong Duc Bui*

## Abstract

We prove that there is a straight-line orthogonal drawing of the complete ternary tree with $n$ nodes in a grid with area $O(n^{1.032})$, improving the best-known bound $O(n^{1.118})$ by Ali [1]. In the special case of drawings satisfying the subtree separation property, we also prove an almost-matching lower bound $\Omega(n^{1.031})$ of this area, resolving a conjecture posed by Covella, Frati and Patrignani [3].

## 1 Introduction

We consider the problem of embedding a tree into a grid. Given a tree $T$ and a grid $G$, an embedding maps each vertex $v$ of $T$ to a distinct vertex $v'$ of $G$, and each edge $uv$ of $T$ to a polyline $u'v'$ of $G$, such that no two polylines intersect except at endpoints.

We call such an embedding *orthogonal*, because all grid lines are either horizontal or vertical. If, furthermore, the embedding satisfies that every edge $uv$ gets mapped to either all horizontal segments or all vertical segments, then we call such an embedding *straight-line orthogonal*. In the literature, straight-line orthogonal embeddings in a grid are also called straight-line orthogonal *drawings*.

Determining whether a straight-line orthogonal drawing of a tree exists is simple—each node of the grid has degree at most 4, so a necessary condition is that each node of the tree has degree at most 4, and it can be shown that this condition is also sufficient. As such, most research on drawings of trees is concerned with minimizing the *area* of the grid, where the area is defined as the number of nodes of the grid.

There has been much research on this problem, see Table 1 for a summary. If the drawing is not required to be straight-line (only orthogonal), [6] proves that there exists an embedding with area $O(n)$ for all embeddable trees, where $n$ is the number of nodes in the tree—this is asymptotically optimal, because the area must be at least $n$. When the drawing is required to be straight-line, the complete binary tree can be drawn in $O(n)$ area by [4], and it is proven in [2] that any binary tree can be drawn in $n \cdot 2^{O(\log^* n)}$ area, which is almost linear.

| | Straight-line | Upper bound | Ref. |
|---|---|---|---|
| Comp. Binary | ✓ | $O(n)$ | [4] |
| Binary | ✓ | $n \cdot 2^{O(\log^* n)}$ | [2] |
| Comp. Ternary | ✓ | $O(n^{1.118})$ | [1] |
| Ternary | ✓ | $O(n^{1.576})$ | [3] |
| Any | | $O(n)$ | [6] |

Table 1: Summary of existing works on tree drawings.

For the ternary case however, the known bounds are less tight. Prior to our work, the best known upper bound for the complete ternary tree is $O(n^{1.118})$ [1], improving upon an existing bound $O(n^{1.262})$ [5]. The best known upper bound for an arbitrary ternary tree is $O(n^{1.576})$ [3].

In this article, we study straight-line orthogonal drawings of the complete ternary tree, and improve the upper bound of the minimum area from $O(n^{1.118})$ to $O(n^{1.032})$. Our method is based on the analysis in [3] of drawings satisfying the subtree separation property. Drawings with this property are more easily analyzed.

We also improve the lower bound of the area needed in the special case of drawings satisfying the subtree separation property to $\Omega(n^{1.031})$. This is the first nontrivial lower bound on the area, with the trivial lower bound being $\Omega(n)$.

This article is organized as follows. In Section 2, we formally define the notations being used. In Section 3, we show the result of a numerical experiment that motivates the proof. In Section 4, we explain the general proof strategy, and prove a weaker upper bound $O(n^{1.051})$ for demonstration. In Section 5, we use a very similar proof strategy to prove the lower bound $\Omega(n^{1.031})$. Finally, in Section 6, we describe our numerical algorithm to provide a certificate of the upper bound $O(n^{1.032})$.

## 2 Definitions

We define the notation for the complete ternary tree following [3].

**Definition 1** *For each positive integer $l$, let $T_l$ be the rooted complete ternary tree with $l$ layers—that is, each non-leaf node has exactly 3 children, and each root-to-leaf path has exactly $l$ nodes.*

*buihd@u.nus.edu

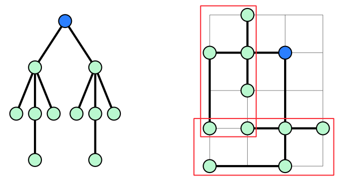

Figure 1: Example of a straight-line orthogonal drawing that does not satisfy the subtree separation property. The tree (left panel, root marked in blue) is embedded in a $4 \times 5$ grid (right panel), and the two subtrees rooted at the two children of the root have intersecting bounding rectangles.

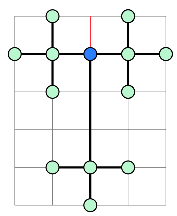

Figure 2: Illustration for $T_3 \subseteq (5, 6)$.

With this definition, $T_1$ has 1 node, $T_2$ has 4 nodes, etc.

We have defined straight-line orthogonal drawings of a tree in the introduction. Now we will formally define the subtree separation property.

**Definition 2** *A drawing is said to satisfy the subtree separation property if, for every nodes a and b of the tree such that the two subtrees rooted at a and b have no nodes in common, the smallest axis-aligned bounding rectangles in the drawing containing all the nodes of these two subtrees have no grid nodes in common.*

See Figure 1 for an illustration of a drawing that does not satisfy the subtree separation property.

We define the following notation for convenience.

**Definition 3** *Given positive integers $l$, $w$ and $h$, where $w$ is odd, write $T_l \subseteq (w, h)$ if there is an orthogonal straight-line drawing of $T_l$ in a grid with width $w$ and height $h$ such that: first, the subtree separation property is satisfied; second, the root of the tree is on the middle vertical grid line; and third, the vertical ray from the root to the top of the grid does not intersect any tree nodes or edges.*

See Figure 2 for an illustration that $T_3 \subseteq (5, 6)$. The red segment in the figure marks the vertical ray from the root to the top of the grid. In order for the drawing to satisfy the third condition of the definition above, no nodes or edges can intersect this red segment.

We define a special class of constructions as follows, which has the advantage of being very easy to analyze. This is a slightly modified form of a 1-2 drawing in [3, Section 3].

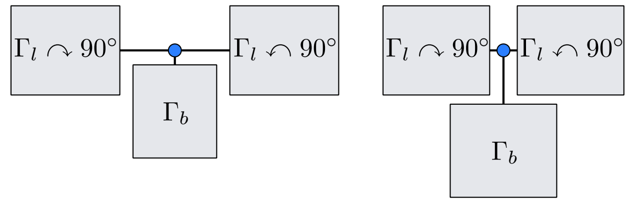


Figure 3: Illustration of Definition 4, with the left panel illustrating construction 1 and the right panel illustrating construction 2.

It is easier to understand the following definition by looking at a picture than reading the description, see Figure 3.

**Definition 4** *We call a straight-line orthogonal drawing of $T_l$ a* symmetric 1-2 drawing *if the following conditions are satisfied. For $l = 1$, the only symmetric 1-2 drawing is the unique drawing on the $1 \times 1$ grid. For $l > 1$, let $\Gamma_r$ and $\Gamma_b$ be two symmetric 1-2 drawings of $T_{l-1}$, then:*

- *define a drawing $\Gamma_1$ created by* construction 1 *as follows: put a copy of $\Gamma_b$ below the root at distance 1, put two copies of $\Gamma_r$ rotated 90° to the left and right of the root at the minimum distance such that the subtree separation property is satisfied.*
- *define a drawing $\Gamma_2$ created by* construction 2 *as follows: put two copies of $\Gamma_r$ rotated 90° to the left and right of the root at distance 1 from the root, then put a copy of $\Gamma_b$ below the root at the minimum distance such that the subtree separation property is satisfied.*

From the definition, we get the following lemma, which also explains the name.

**Lemma 5** *All symmetric 1-2 drawings have odd width, and are vertically symmetric. Furthermore, let the size of $\Gamma_r$ be $(w_r, h_r)$ and the size of $\Gamma_b$ be $(w_b, h_b)$, then the size of $\Gamma_1$ is $\mathbf{c}_1(w_r, h_r, w_b, h_b)$ and the size of $\Gamma_2$ is $\mathbf{c}_2(w_r, h_r, w_b, h_b)$.*

Where we define two functions $\mathbf{c}_1, \mathbf{c}_2 \colon \mathbb{R}^4 \to \mathbb{R}^2$ by

$$\mathbf{c}_1(w_r, h_r, w_b, h_b) = \left(2h_r + w_b, \frac{w_r}{2} + \max\left(\frac{w_r}{2}, h_b + \frac{1}{2}\right)\right),$$

$$\mathbf{c}_2(w_r, h_r, w_b, h_b) = \big(\max(2h_r + 1, w_b), w_r + h_b\big).$$

As we have mentioned, the symmetric 1-2 drawings are very easy to analyze. In particular, we can compute *all* grid sizes $(w, h)$ such that $T_l \subseteq (w, h)$. The algorithm to compute these grid sizes was given in [3, Lemma 5]:

**Lemma 6** *For a fixed $l$, the Pareto-optimal pairs $(w, h)$ at level $l$ can be computed in time polynomial in the number of nodes of $T_l$.*

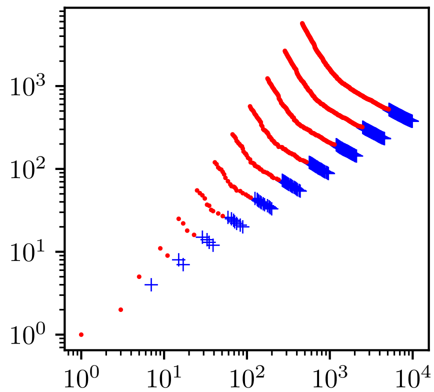


Figure 4: A scatterplot of all Pareto-optimal grid sizes. Blue crosses denote construction 1, red dots denote construction 2. We see from the figure that construction 1 is preferred when $w \gg h$.

We should explain what Pareto-optimal pairs mean in this lemma. Because if $w \le w'$ and $h \le h'$ then $T_l \subseteq (w,h) \implies T_l \subseteq (w',h')$, it suffices to consider for each $l$ the pairs $(w,h)$ such that $T_l \subseteq (w,h)$ and there exist no pair $(w',h')$ such that $w' \le w$, $h' \le h$, $w' \cdot h' < w \cdot h$, and $T_l \subseteq (w',h')$. We call these pairs *Pareto-optimal* at level $l$.

Apart from being easy to analyze, the symmetric 1-2 drawings additionally satisfy the following properties, which is proven in [3, Lemma 3].

**Lemma 7** *Given any straight-line orthogonal drawing $\Gamma$ of the complete ternary tree $T_l$, there exists a rotation of a symmetric 1-2 drawing $\Gamma'$ whose both width and height are no more than those of $\Gamma$.*

We allow 90° rotations, which swaps the width and the height.

## 3 Motivation: The Pattern of the Pareto-optimal Grid Sizes

We compute the Pareto-optimal grid sizes for small values of $l$:

- When $l = 1$, the only pair is $(1,1)$.
- When $l = 2$, the only pair is $(3,2)$.
- When $l = 3$, there is a pair $(5,5)$ corresponding to construction 2, and a pair $(7,4)$ corresponding to construction 1.

We make a scatterplot for all the pairs for each value of $l$. The result is shown in Figure 4, where both $x$-axis and $y$-axis use a logarithmic scale.

From the figure, the pattern is obvious. Our goal is thus to prove that the pattern continues indefinitely.

In order to do so, we need to look at how these grid sizes were computed—the set of Pareto-optimal grid sizes at level $l$ is computed only from the Pareto-optimal grid sizes at level $l-1$, independent of what happens at earlier levels. As such, our proof will be inductive—assume the Pareto-optimal grid sizes at level $l-1$ satisfy some bound, we prove the Pareto-optimal grid sizes at level $l$ satisfy another bound.

In order to formalize these concepts, we make the following definitions.

**Definition 8 ($\le$-relation for grid sizes)** *Let $w$, $h$, $w'$, $h'$ be real numbers. We say $(w,h) \le (w',h')$ if $w \le w'$ and $h \le h'$. Similarly, $(w,h) \ge (w',h')$ if $w \ge w'$ and $h \ge h'$.*

**Definition 9** *For any set $A \subseteq \mathbb{R}^2$, define the* upper-closure *$C(A) \subseteq \mathbb{R}^2$ to be $C(A) = \{(w,h) \in \mathbb{R}^2 \mid \exists (w',h') \in A, (w',h') \le (w,h)\}$. We say a set $A$ is* upper-closed *if $A \subseteq \mathbb{R}^2$ and $C(A) = A$.*

**Definition 10** *For each $l \ge 1$, define the set $E_l$ to be all pairs $(w,h) \in \mathbb{Z}^2$ such that $T_l \subseteq (w,h)$. Define $S_l = C(E_l)$.*

So for example, at $l = 2$, $E_l$ consists of all pairs of integers $(w,h)$ such that $w$ is odd, $w \ge 3$ and $h \ge 2$, while $S_l$ consists of all pairs of reals $(w,h)$ such that $w \ge 3$ and $h \ge 2$. We see that $S_l$ is the natural extension of $E_l$ to the domain of all reals.

**Definition 11** *For any upper-closed set $A \subseteq \mathbb{R}^2$ and real number $\delta$, define the* shift *of $A$ by $\delta$ to be $\Delta(A,\delta) = \{(w \cdot \exp\delta, h \cdot \exp\delta) \mid (w,h) \in A\}$.*

**Definition 12** *For any upper-closed set $A \subseteq \mathbb{R}^2$, define the* advance *of $A$ to be*

$$N(A) = C\Big(\big\{\mathbf{c}_i(w_r,h_r,w_b,h_b) \mid (w_r,h_r) \in A, (w_b,h_b) \in A, i \in \{1,2\}\big\}\Big).$$

The functions $\mathbf{c}_1$ and $\mathbf{c}_2$ were introduced for Lemma 5. As such, we get the following:

**Lemma 13** *For each $l \ge 1$, $N(S_l) = S_{l+1}$.*

Let $A$ and $B$ be upper-closed sets. We say $A \le B$ if the "boundary" of $A$ is below the "boundary" of $B$. For example, if $A = \{(x,y) \mid x \in \mathbb{R}, y \ge 1\}$ and $B = \{(x,y) \mid x \in \mathbb{R}, y \ge 2\}$, then visually, the boundary of $A$ is below the boundary of $B$. The formal definition is:

**Definition 14** *For two upper-closed sets $A$ and $B$, we say $A \le B$ if $A \supseteq B$, and $A \ge B$ if $A \subseteq B$.*

Even though the direction of $\le$ appears reversed compared to the $\subseteq$, by the explanation above, this is the intuitively correct direction. It is also consistent with the $\le$ notation defined for grid sizes earlier—for $(w,h)$ and $(w',h') \in \mathbb{R}^2$, $(w,h) \le (w',h')$ if and only if $C(\{(w,h)\}) \le C(\{(w',h')\})$.

## 4 Upper Bound: Preliminary

Using these definitions, we explain how the proof of the upper bound proceeds. As previously explained, it will use induction, where $\delta$ is a positive real constant and $T \subseteq \mathbb{R}^2$ is a fixed set.[1]

- Base case: $S_{l_0} \leq T$.
- Induction step: If $S_l \leq \Delta(T, d)$ for any $d \geq 0$, then $S_{l+1} \leq \Delta(T, d + \delta)$.

By induction, we get $S_l \leq \Delta(T, \delta \cdot (l - l_0))$ for all $l \geq l_0$. For a suitable set $T$, this implies the complete ternary tree $T_l$ can be embedded in a grid with area $O(e^{2\delta l}) = O(n^{2\delta/\log 3})$ where $n$ is the number of nodes in the tree $T_l$.

For example, we may use $T = S_{18}$. Then the base case is trivially satisfied for $l_0 = 18$. The constant $\delta$ used is $\log(63761/35808)$, which is in fact the smallest $\delta$ such that $S_{19} \leq \Delta(S_{18}, \delta)$. To compute this value of $\delta$, we used the algorithm described in [3, Lemma 5] to compute both $S_{18}$ and $S_{19}$ explicitly. We also have $2\delta/\log 3 < 1.051$, therefore $T_l$ can be embedded in a grid with area $O(n^{1.051})$.

Now we prove that the induction step holds.

**Lemma 15** *Let $d \geq 0$. If $N(T) \leq U$, then $N(\Delta(T, d)) \leq \Delta(U, d)$.*

**Proof.** Expanding out the definitions, it suffices to show the following. For all $i \in \{1, 2\}$, $d \geq 0$, positive reals $w_r$, $h_r$, $w_b$, $h_b$, let $(w, h) = \mathbf{c}_i(w_r, h_r, w_b, h_b)$, then we need to prove

$$\mathbf{c}_i(w_r \cdot e^d, h_r \cdot e^d, w_b \cdot e^d, h_b \cdot e^d) \leq (w \cdot e^d, h \cdot e^d).$$

So for example, when $i = 2$, for the first component, we need to prove

$$\max(2h_r \cdot e^d + 1, w_b \cdot e^d) \leq \max(2h_r + 1, w_b) \cdot e^d.$$

Since $d \geq 0$, $e^d \geq 1$ and $(2h_r + 1) \cdot e^d \geq 2h_r \cdot e^d + 1$. Other cases are omitted because they are similar. ☐

Using this, the induction step can be proven. Since $N(T) \leq \Delta(T, \delta)$, we get

$$N(\Delta(T, d)) \leq \Delta(\Delta(T, \delta), d) = \Delta(T, \delta + d).$$

The induction hypothesis gives us $S_l \leq \Delta(T, d)$, so

$$N(S_l) = S_{l+1} \leq N(\Delta(T, d)).$$

Combining the two inequalities, we get $S_{l+1} \leq \Delta(T, \delta + d)$ as desired.

To complete the proof, we just need the following.

[1]We apologize for using $T$ to denote a tree instead of a set in the introduction.

**Lemma 16** *Let $T \subseteq \mathbb{R}^2$ be any non-empty upper-closed set, and $l_0$ be a fixed integer. If $S_l \leq \Delta(T, \delta \cdot (l - l_0))$ for all positive integers $l \geq l_0$, then the area of the smallest grid that $T_l$ can be embedded in is $O(n^{2\delta/\log 3})$, where $n \in \Theta(3^l)$ is the number of nodes in $T_l$.*

**Proof.** Pick an arbitrary fixed element $(w, h) \in T$. By definition, $S_l \leq \Delta(T, \delta \cdot (l - l_0))$, so $(w \cdot e^{\delta \cdot (l - l_0)}, h \cdot e^{\delta \cdot (l - l_0)}) \in S_l$. As such, there is a grid with area no more than $w \cdot e^{\delta \cdot (l - l_0)} \cdot h \cdot e^{\delta \cdot (l - l_0)}$ that $T_l$ can be embedded in, this value is $O(n^{2\delta/\log 3})$ as needed. ☐

## 5 Lower Bound

Similarly, we will fix a set $S$, a positive real $\delta$, and a positive integer $l_0$, and prove:

- Base case: $S_{l_0} \geq S$;
- Induction step: For any $l$ and $d$, if $S_l \geq \Delta(S, d)$, then $S_{l+1} \geq \Delta(S, d + \delta)$.

As such, for every $l \geq l_0$, we get $S_l \geq \Delta(S, (l - l_0) \cdot \delta)$. For a suitable initial set $S$, this implies the smallest area of a grid that $T_l$ can be embedded in is $\Omega(\exp(2\delta \cdot l))$.

This time however, the analog of Lemma 15 would be the following (we do not need to use this in the article, as such it is not proved):

> Let $d \leq 0$. If $N(S) \geq U$, then $N(\Delta(S, d)) \geq \Delta(U, d)$.

Note that $d \leq 0$. This means inequalities can only be shifted "backward", not forward. As such, we would need to conceptually define a set $S$ "at infinity", then shift it backwards. To formalize it, we give the following definition.

**Definition 17** *For any upper-closed set $A \subseteq \mathbb{R}^2$, define the* advance at infinity *of $A$ to be*

$$N^\infty(A) = C\Big(\big\{(2h_r + w_b, \tfrac{w_r}{2} + \max(\tfrac{w_r}{2}, h_b)) \mid (w_r, h_r) \in A, (w_b, h_b) \in A\big\} \cup \big\{(\max(2h_r, w_b), w_r + h_b) \mid (w_r, h_r) \in A, (w_b, h_b) \in A\big\}\Big).$$

This should be thought of as $\lim_{d \to +\infty} \Delta(N(\Delta(A, d)), -d)$—shift $A$ to "infinity", advance it, then shift it back.

Also note that $N^\infty$ is invariant under $\Delta$-shifting—formally, for any real $d$, $N^\infty(\Delta(S, d)) = \Delta(N^\infty(S), d)$.

We get the following:

**Lemma 18** $N(S) \geq N^\infty(S)$. *Therefore, if $N^\infty(S) \geq U$, then $N(S) \geq U$.*

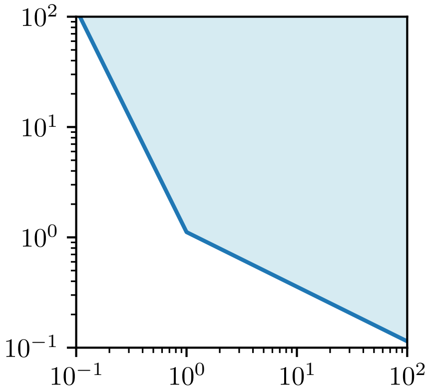


Figure 5: Illustration of the set $S$ used in the lower bound proof.

Using reasoning similar to the previous section, the induction proceeds as follows. Assume the set $S$ satisfies $N^\infty(S) \geq \Delta(S, \delta)$. Then assume the base case $S_1 \geq S$ holds, the induction step can be proven as follows. Using Lemma 18,

$$N(\Delta(S, d)) \geq N^\infty(\Delta(S, d)) = \Delta(N^\infty(S), d) \geq \Delta(S, d + \delta).$$

From the induction hypothesis, $S_l \geq \Delta(S, d)$, so

$$S_{l+1} = N(S_l) \geq N(\Delta(S, d)).$$

Combining the two inequalities, we get $S_{l+1} \geq \Delta(S, d + \delta)$ as needed.

Now, the only remaining challenge is to construct such a set $S$.

**Definition 19** *Fix constants $\sigma > 1$ and $\varepsilon \in \mathbb{R}$. Define $S \subseteq \mathbb{R}^2$ to be the upper-closure of the set of points $\{\exp(\omega, \max(\frac{-\omega}{\sigma}, -\omega \cdot \sigma) + \varepsilon) \mid \omega \in \mathbb{R}\}$. Here, we write $\exp(\omega, \eta)$ to denote the pair $(\exp \omega, \exp \eta)$.*

**Lemma 20** *For constants $\delta = 0.5667$, $\varepsilon = 0.10995$ and $\sigma = 2.01979$, we have $N^\infty(S) \geq \Delta(S, \delta)$.*

The proof will be deferred for later. In the log-log plot, this set $S$ is an unbounded polygon as illustrated in Figure 5, with the vertex at $\exp(0, \varepsilon)$. Using this lemma, we get the following:

**Theorem 21** *Set $\delta$ as above. If $T_l \subseteq (w, h)$, then $w \cdot h \in \Omega(n^{2\delta/\log 3}) \geq \Omega(n^{1.031})$, where $n \in \Theta(3^l)$ is the number of nodes in $T_l$.*

Note that by our definition of $\leq$, the theorem only lower bounds the area of embeddings satisfying the subtree separation property.

**Proof.** Note that for $l_0 = 2$ then $S_2 \geq S$.

Apply induction by the plan described above, we get $S_l \geq \Delta(S, (l - l_0) \cdot \delta)$ for all $l \geq 2$.

Note that for every $(w, h) \in S$ then $w \cdot h \geq \exp \varepsilon$, therefore for every $(w, h) \in \Delta(S, (l - l_0) \cdot \delta)$ then $w \cdot h \geq \exp(2(l - l_0)\delta + \varepsilon)$, so we are done. □

Now we prove Lemma 20.

**Proof.** Pick $(w, h) \in N^\infty(S)$. Define $w_r$, $h_r$, $w_b$, $h_b$ as in Definition 17, then $(w_r, h_r) \in S$ and $(w_b, h_b) \in S$.

Expanding out these conditions, we get that the assumptions are, for both $\tau \in \{\sigma, \frac{1}{\sigma}\}$:

$$\log h_r \geq \varepsilon - \log w_r \cdot \tau,$$
$$\log h_b \geq \varepsilon - \log w_b \cdot \tau.$$

We need to prove $(w, h) \in \Delta(S, \delta)$. This is equivalent to the following statement: for both $\tau \in \{\sigma, \frac{1}{\sigma}\}$,

$$\log h \geq \delta + \varepsilon + (\delta - \log w) \cdot \tau.$$

In the first case (construction 1), it suffices for us to prove for both $\tau \in \{\sigma, \frac{1}{\sigma}\}$:

$$\log\left(\tfrac{w_r}{2} + h_b\right) \geq \delta + \varepsilon + (\delta - \log(2h_r + w_b)) \cdot \tau.$$

For any positive reals $a$ and $b$, $\log(a + b) \geq \max(\log a, \log b)$. Define the softmax function $\mathrm{sm}(a, b) = \log(\exp a + \exp b)$, then the left-hand side is $> \log(w_r + h_b) = \mathrm{sm}(\log w_r, \log h_b)$.

Define $\omega_i = \log w_i$, $\eta_i = \log h_i$ for $i \in \{r, b\}$. Then we need to prove

$$\mathrm{sm}(\omega_r - \log 2, \eta_b) \geq \delta + \varepsilon + (\delta - \mathrm{sm}(\eta_r + \log 2, \omega_b)) \cdot \tau.$$

With this new notation, for all $\tau \in \{\sigma, \frac{1}{\sigma}\}$ and $i \in \{r, b\}$, then

$$\eta_i \geq \varepsilon - \omega_i \cdot \tau.$$

Equivalently,

$$\omega_i \geq (\varepsilon - \eta_i) \cdot \tau.$$

Thus, we just need to prove

$$\mathrm{sm}((\varepsilon - \eta_r) \cdot \tau - \log 2, \varepsilon - \omega_b \cdot \tau) \geq \delta + \varepsilon + (\delta - \mathrm{sm}(\eta_r + \log 2, \omega_b))\tau.$$

Set $\varphi = \eta_r - \omega_b$, since the sm function satisfies $\mathrm{sm}(a + d, b + d) = \mathrm{sm}(a, b) + d$ for all $a$, $b$, $d \in \mathbb{R}$, this simplifies to

$$\mathrm{sm}((\varepsilon - \varphi)\tau - \log 2, \varepsilon) \geq (1 + \tau)\delta + \varepsilon - \mathrm{sm}(\varphi + \log 2, 0)\tau.$$

Equivalently, since $1 + \tau > 0$,

$$\delta \leq \frac{\mathrm{sm}((\varepsilon - \varphi)\tau - \log 2, \varepsilon) - \varepsilon + \mathrm{sm}(\varphi + \log 2, 0)\tau}{1 + \tau}.$$

The right hand side only contains one variable $\varphi$. We define the function $f_{0,\tau}(\varphi)$ to be the right hand side, and we wish to compute the minimum of $f_{0,\tau}$. Notice that $f'_{0,\tau}(\varphi) = 0$ has an unique solution

$$\varphi = \frac{(\tau - 1)\varepsilon - \log 4}{1 + \tau}$$

and this can be shown to be the global minimum of $f_{0,\tau}$. At this point, the value of $f_{0,\tau}$ is $\geq \delta$ for both $\tau = \sigma$ and $\tau = \frac{1}{\sigma}$, so we are done.

In the second case (construction 2), we need to prove:

$$\log(w_r + h_b) \geq \delta + \varepsilon + (\delta - \log(\max(w_b, 2h_r))) \cdot \tau.$$

This simplifies to

$$\mathrm{sm}(\omega_r, \eta_b) \geq \delta \cdot (1 + \tau) + \varepsilon - \max(\omega_b, \eta_r + \log 2) \cdot \tau.$$

Doing exactly as above, we just need to prove

$$\mathrm{sm}((\tau - 1)\varepsilon - \eta_r\tau, -\omega_b\tau) \geq \delta \cdot (1 + \tau) - \max(\omega_b, \eta_r + \log 2) \cdot \tau.$$

Set $\varphi = \eta_r - \omega_b$ again, this further simplifies to

$$\delta \leq \frac{\mathrm{sm}(0, (\tau - 1)\varepsilon - \varphi\tau) + \max(0, \varphi + \log 2) \cdot \tau}{1 + \tau}.$$

Set $f_{1,\tau}(\varphi) = \frac{\mathrm{sm}(0,(\tau-1)\varepsilon-\varphi\tau)}{1+\tau}$ and $f_{2,\tau}(\varphi) = \frac{\mathrm{sm}(0,(\tau-1)\varepsilon-\varphi\tau)+(\varphi+\log 2)\cdot\tau}{1+\tau}$, notice that $f_{1,\tau}$ is decreasing, $f_{2,\tau}$ is increasing, and the equation $f_{1,\tau}(\varphi) = f_{2,\tau}(\varphi)$ has a unique solution $\varphi = -\log 2$, at this point

$$f_{1,\tau}(\varphi) = f_{2,\tau}(\varphi) = \frac{\mathrm{sm}(0, (\tau - 1)\varepsilon + \tau \log 2)}{1 + \tau}.$$

As such, $\max(f_{1,\tau}(\varphi), f_{2,\tau}(\varphi))$ has a global minimum at this point. For both $\tau \in \{\sigma, \frac{1}{\sigma}\}$, this value is $\geq \delta$. □

We are unable to obtain a closed-form formula for these constants. However, it can be efficiently computed to arbitrary precision using software such as Mathematica—namely, find the values of $\sigma$, $\varepsilon$ and $\delta$ such that

$$\delta = f_{1,\sigma}(-\log 2) = f_{1,1/\sigma}(-\log 2) = f_{0,1/\sigma}\Big(\frac{(1/\sigma - 1)\varepsilon - \log 4}{1 + 1/\sigma}\Big), \quad (1)$$

then it can be confirmed that $f_{0,\sigma}\big(\frac{(\sigma-1)\varepsilon-\log 4}{1+\sigma}\big) > \delta$.

## 6 Numerical Proof for Improved Upper Bound

In Section 4, we improved the upper bound on the minimum area required for a straight-line orthogonal drawing of $T_l$. In order to do so, we used a certain upper-closed set $T$ satisfying $N(T) \leq \Delta(T, \delta)$ for a constant $\delta < 0.577$. Specifically, we used $T = S_{18}$.

If we use a different set $T$ such that $N(T) \leq \Delta(T, \delta')$ for a smaller constant $\delta'$, we would be able to improve the upper bound accordingly. In order to simplify the analysis, we use the following.

**Lemma 22** *Fix an upper-closed set $T$ such that $C(\{1, 1\}) \leq T$, and a constant $\delta > 0$. If $N^\infty(T) \leq \Delta(T, \delta)$, then for all $\varepsilon > 0$, there exists sufficiently large $d > 0$ such that $N(T') \leq \Delta(T', \delta + \varepsilon)$ for $T' = \Delta(T, d)$.*

**Proof.** Unrolling the definition, we need to prove that with notation as above, for every $(w, h) \in \Delta(T', \delta + \varepsilon)$, then $(w, h) \in N(T')$.

The statement $(w, h) \in \Delta(T', \delta + \varepsilon)$ is equivalent to $w = \exp(d+\delta+\varepsilon)w'$, $h = \exp(d+\delta+\varepsilon)h'$ for $(w', h') \in T$.

By assumption, $N^\infty(T) \leq \Delta(T, \delta)$, so $N^\infty(\Delta(T, d + \varepsilon)) \leq \Delta(T, d+\delta+\varepsilon)$, which means $(w, h) \in N^\infty(\Delta(T, d+\varepsilon))$. Expanding out the definition of $N^\infty$, this means there exists $(w_r, h_r)$ and $(w_b, h_b) \in \Delta(T, d + \varepsilon)$ such that either

$$w = 2h_r + w_b \text{ and } h = \frac{w_r}{2} + \max\Big(\frac{w_r}{2}, h_b\Big) \quad (2)$$

or

$$w = \max(2h_r, w_b) \text{ and } h = w_r + h_b. \quad (3)$$

The statement we need to prove is $(w, h) \in N(T')$. With $(w_r, h_r)$ and $(w_b, h_b)$ as above, we get

$$(w_r/\exp\varepsilon, h_r/\exp\varepsilon), (w_b/\exp\varepsilon, h_b/\exp\varepsilon) \in T'. \quad (4)$$

In the first case where Eq. 2 holds, we will use Eq. 4 to get:

$$\Big(\frac{2h_r + w_b}{\exp\varepsilon}, \frac{w_r}{2\exp\varepsilon} + \max\Big(\frac{w_r}{2\exp\varepsilon}, \frac{h_b}{\exp\varepsilon} + \frac{1}{2}\Big)\Big) \in N(T').$$

We will show that it is possible to pick $d$ large enough such that both $w \geq \frac{2h_r+w_b}{\exp\varepsilon}$ and $h \geq \frac{w_r}{2\exp\varepsilon} + \max(\frac{w_r}{2\exp\varepsilon}, \frac{h_b}{\exp\varepsilon} + \frac{1}{2})$ holds. Assume otherwise. Since $\varepsilon > 0$, $w \geq \frac{2h_r+w_b}{\exp\varepsilon}$ always, so $\frac{w_r}{2} + \max(\frac{w_r}{2}, h_b) < \frac{w_r}{2\exp\varepsilon} + \max(\frac{w_r}{2\exp\varepsilon}, \frac{h_b}{\exp\varepsilon} + \frac{1}{2})$, so $\max(\frac{w_r}{2}, h_b) < \max(\frac{w_r}{2\exp\varepsilon}, \frac{h_b}{\exp\varepsilon} + \frac{1}{2})$. Therefore $\frac{w_r}{2} < \frac{h_b}{\exp\varepsilon} + \frac{1}{2}$, so $\max(\frac{w_r}{2\exp\varepsilon}, \frac{h_b}{\exp\varepsilon} + \frac{1}{2}) = \frac{h_b}{\exp\varepsilon} + \frac{1}{2}$, so $h_b < \frac{h_b}{\exp\varepsilon} + \frac{1}{2} \iff h_b < \frac{1}{2(1-\exp(-\varepsilon))}$. Since $(w_b, h_b) \in \Delta(T, d + \varepsilon)$ and $C(\{1, 1\}) \leq T$, by picking $d$ large enough, we can make $h_b < \frac{1}{2(1-\exp(-\varepsilon))}$ impossible. Note that $d$ still only depends on $\varepsilon$ and not on $h_b$.

When Eq. 3 holds instead, we use Eq. 4 to get

$$\Big(\max\Big(\frac{2h_r}{\exp\varepsilon} + 1, \frac{w_b}{\exp\varepsilon}\Big), \frac{w_r + h_b}{\exp\varepsilon}\Big) \in N(T').$$

Proceed similarly. □

We now explain how the set $T$ was found. Consider the set $S$ in Section 5. With values of $\sigma$, $\varepsilon$ and $\delta$ satisfying Eq. 1 exactly, we get $S \leq \Delta(N^\infty(S), -\delta)$. Define the operator $P(S) = \Delta(N^\infty(S), -\delta)$ and let $P^k(S)$ be $P$ applied $k$ times on $S$, then we get $S \leq P(S)$, which means $P^k(S) \leq P^{k+1}(S)$ for all $k$, and we conjecture the following based on numerical evidence:

**Conjecture 1** $\lim_{k\to\infty} P^k(S) = \{(w, h) \mid h \geq f(w)\}$ *for an analytic function* $f$ *such that* $f(\exp\omega) - (\varepsilon - \omega \cdot \sigma) \to 0$ *as* $\omega \to -\infty$ *and* $f(\exp\omega) - (\varepsilon - \omega/\sigma) \to 0$ *as* $\omega \to +\infty$. *Furthermore, let* $P^\infty(S)$ *denote that limiting set, then* $P(P^\infty(S)) = P^\infty(S)$.

Such a function $f$ must satisfy a certain functional equation; however, we are unable to solve it or prove that the solution exists. If the conjecture holds, pick $T = \Delta(P^\infty(S), d) \cap C(\{w, h\})$ for some large $d$ and suitable $w$ and $h$, we get $N^\infty(T) \leq \Delta(T, \delta)$, which would have implied the area is $\in \widetilde{\Theta}(n^{2\delta/\log 3})$. However, we are only able to find a numerical approximation of $T$ that gives the bound $O(n^{1.032})$.

The algorithm to compute the certificate $T$ for the upper bound computes $P^k(S)$ for a sufficiently large number $k$, with some approximations and linear interpolations to make the time complexity manageable, and we only iterate on $\mathbf{c}_2$ in the initial iterations, which heuristically have better numerical stability.

## 7 Conclusion

In this article, we improve the upper bound on the minimum area required for a straight-line orthogonal drawing of the complete ternary tree, and prove an almost-matching lower bound in the special case of drawings satisfying the subtree separation property.

There are still several interesting open questions that need further research, namely whether there is a non-trivial lower bound when the drawing is not required to satisfy the subtree separation property, and whether the constant can be determined analytically to prove that our lower bound is tight.

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