On t-fold Totally-Concave Polyominoes

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Abstract

A *t-fold totally concave polyomino* (t-TCP) is an edge-wise connected collection of square cells with t or more gaps in every row and column. We prove that the minimum area of the smallest possible t-TCP is 21 for $t = 1$, 50 for $t = 2$, and $6(t + 1)^2 - 1$ for $t > 2$. Answering a previous conjecture on the affirmative, we prove that the t-TCP counting sequence has the same leading exponential order as all polyominoes, from which we prove that the ratio of successive terms converges.

1 Introduction

A polyomino is an edge-wise connected collection of unit squares in the plane. That is, given a connected subgraph G of the square lattice (with nodes at integer coordinates), the polyomino determined by G is $P_G := \bigcup_{(x,y)\in G} [x, x + 1] \times [y, y + 1]$. To consider only one translate of each polyomino, we use the convention that every polyomino P satisfies $P \subset [0, \infty) \times [0, \infty)$, $P \cap (\{0\} \times \mathbb{R}) \neq \emptyset$, and $P \cap (\mathbb{R} \times \{0\}) \neq \emptyset$. A row or a column ξ of a polyomino has a *gap* if ξ contains at least two maximal sequences of consecutive cells; likewise, ξ has t gaps if it consists of at least $t+1$ maximal sequences of consecutive cells. *Totally Concave Polyominoes (TCPs)* are those polyominoes in which every row and every column of cells has at least one "gap." Figure 1 shows a non-TCP, while Figure 2(a) shows a similar TCP. The difference between the two polyominoes is that the former one is missing the bottom left cell. Note that it is sometimes useful to consider a polyomino just as a subgraph of the square lattice, and sometimes equivalently as a collection of square cells. Throughout the paper, our drawings will show both, when convenient. TCPs were introduced in the Handbook of Discrete and Computational Geometry [6] as an extremal opposite of *convex polyominoes*, a much more extensively studied set for which an asymptotic formula is known [7].

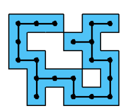

Figure 1: Not TCP.

TCPs were first investigated in-depth recently in reference [2]. There,

i. The minimum possible area for a TCP was proved to be 21;

ii. The number of area-n TCPs, $\kappa(n)$, was evaluated for $21 \leq n \leq 35$; and

iii. The TCP growth constant, $\lambda_\kappa := \lim_{n\to\infty} \sqrt[n]{\kappa(n)}$ was shown to exist, bounded from below, and conjectured to be equal to $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$, the growth constant of all polyominoes.

In this paper, for each positive t, we consider t-fold TCPs (t-TCPs): those with at least t gaps in every row and column. Figure 2 shows a few examples which are in fact of the minimum possible sizes (see Section 3). We generalize the result on minimal examples to t-TCPs, answer the mentioned conjecture, and prove the existence of another important limit by strengthening an established technique.

The symbols A_n, κ_n, and $\kappa_{t,n}$ will denote the sets of area-n polyominoes, TCPs, and t-TCPs, respectively, while $A(n)$, $\kappa(n)$, and $\kappa_t(n)$ will denote the number of these objects. In addition, we use the following lexicographic order of cells on the square lattice.

Definition 1 *(Lexicographic Order) Given two cells on the square lattice, $c_1 = [x_1, x_1+1]\times[y_1, y_1+1]$ and $c_2 = [x_2, x_2+1]\times[y_2, y_2+1]$, we say that c_1 is lexicographically smaller than c_2 if $x_1 < x_2$, or if $x_1 = x_2$ and $y_1 < y_2$.*

2 A Physical Context

In statistical physics, it is generally believed that typical polyominoes (and, more generally, lattice trees and lattice animals, which are models of branched polymers) display a fractal geometry in the limit as their size gets large. This is supported by non-rigorous scaling theory as well as by simulations. However, almost nothing has been proven rigorously about the asymptotic geometry of these objects (except in high dimensional space, which here means above eight dimensions). Fractal behavior in two dimensions would imply in particular that a vertical or horizontal line intersecting a large polyomino would typically have gaps on all length scales. This seems hard to prove, but a simpler task would be to show that, for each positive integer t, there is a reasonable probability that most lines intersecting a

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sufficiently-large polyomino would have at least t gaps. Our result upgrades "most" to "every," but replaces "reasonable probability" with "probability that is not exponentially small". (In fact, it is only polynomially small under the standard belief that $A(n)$ is asymptotically proportional to $n^{-\theta}\lambda^n$ for some constant $\theta > 0$.) The probability for a given size, of course, is the ratio of t-TCPs to all polyominoes of that size. We conjecture that this probability is bounded away from 0 as the size tends to infinity. (A simple argument proves that it is bounded away from 1.) This conjecture seems to be a difficult problem to resolve rigorously, but we view our theorem as a first rigorous corroboration (albeit a mild one) of the behavior that physicists expect.

3 Minimum Area of t-TCPs

We will prove a lower bound on the area of t-TCPs for all t, and construct examples to prove it is tight, which proves the following theorem, our first main result.

Let m_t be the minimum possible area of a t-TCP.

Theorem 1 *$m_1 = 21$, $m_2 = 50$, and $m_t = 6(t+1)^2 - 1$ for all $t > 2$.* □

The theorem above is the combination of lemmata 3 and 4 below.

3.1 Lower Bound

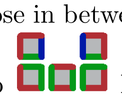
We use the notion of the *minimum bounding box* of a polyomino.

Definition 2 *The* minimum bounding box *of a polyomino P is the least pair of integers (k, ℓ), such that $P \subset [0, k] \times [0, \ell]$. That is, the minimum bounding box of P is contained in any other bounding box of P.*

The relations in Lemma 2 for the $t = 1$ case were first described in reference [2]. What follows is to our knowledge the first treatment of them in the general-t case.

Lemma 2 *For a t-fold TCP with n cells in a (k, ℓ)-bounding box,*

$$(t+1)(k+\ell) - 1 \le n \le k\ell - t \cdot \max\{k, \ell\} - 2t.$$

Proof. By rotating if necessary, we may assume $k \ge \ell$. For the lower bound, partition the edges of the polyomino's cells into *outside*, *inside*, and *hidden* edges, which we will say number o, i, and h, respectively. Outside edges face away from the polyomino, inside edges back into it, and hidden edges are those in between two cells. For example, the U-pentomino  has ten outside edges (red), two inside edges (blue), and eight hidden edges (green). Being the perimeter, $o = 2k + 2\ell$. By the t-TC property, $i \ge 2tk + 2t\ell$. By connectedness, every polyomino has a spanning tree with at least $n - 1$ edges, and so $h \ge 2n - 2$. The lower bound follows from this and the fact that we counted exactly $4n = o + i + h$ edges. For the upper bound, notice that we must remove at least tk cells from $[0, k] \times [1, \ell - 1]$ in order to have k-many t-fold concave columns, and a further t cells from the top and the bottom rows to guarantee their t-fold concavity. Finally, we take in the statement of the lemma the maximum of k and ℓ since their roles can be exchanged. □

These relations restrict the possible areas of t-TCPs in (k, ℓ) bounding boxes rather significantly. We see this by solving an integer non-linear program (NLP) in the general-t case using duality. Guenin *et al.* [8] provide a friendly reference for the techniques used.

Lemma 3 *$m_1 \ge 21$, $m_2 \ge 50$, and $m_t \ge 6(t+1)^2 - 1$ for $t > 2$.*

Proof. Assume, without loss of generality, that $k \ge \ell$. For a t-TCP to exist in a (k, ℓ) bounding box, the lower and upper bounds of Lemma 2 must both hold, *i.e.*, their difference $H(k, \ell) := k\ell - (2t+1)k - (t+1)\ell - 2t + 1$ must be non-negative. Minimizing the lower bound of Lemma 2, we therefore have the integer NLP (1). To solve (1), we will consider two auxiliary NLPs, (2) and (3).

$$\begin{array}{ll} \min & k+\ell \\ \text{s.t.} & H(k,\ell) \ge 0, \\ & k-\ell \ge 0, \text{ and} \\ & k,\ell \in \mathbb{Z}^+. \end{array} \tag{1}$$

$$\begin{array}{ll} \min & k+\ell \\ \text{s.t.} & H(k,\ell) \ge 0, \\ & k-\ell \ge 0, \text{ and} \\ & k,\ell \ge 0. \end{array} \tag{2}$$

$$\begin{array}{ll} \min & k+\ell \\ \text{s.t.} & H(k,\ell) \ge 0, \\ & k-\ell \ge 1, \text{ and} \\ & k,\ell \ge 0 \end{array} \tag{3}$$

First, we solve the NLP (2). Noting the region $\{(k, \ell) : H(k, \ell) \ge 0, k, \ell > 0\}$ is convex, we may define a linear relaxation by a gradient, the LP (4). We also write its dual, the LP (5),

$$\begin{array}{ll} \min & [1\ 1]\,[k\ \ell]^T \\ \text{s.t.} & \begin{bmatrix} 1 & -1 \\ 1 & -\alpha_t \end{bmatrix} \begin{bmatrix} k \\ \ell \end{bmatrix} \ge \begin{bmatrix} 0 \\ \beta_t - \alpha_t\beta_t \end{bmatrix} \\ \text{where} & k,\ell \ge 0 \end{array} \tag{4}$$

$$\begin{array}{ll} \max & [0\ (\beta_t - \alpha_t\beta_t)]\,[x\ y]^T \\ \text{s.t.} & \begin{bmatrix} 1 & 1 \\ -1 & -\alpha_t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \text{where} & x,y \ge 0 \end{array} \tag{5}$$

where the number β_t is such that (β_t, β_t) is the point of intersection of the line $k = \ell$ and the hyperbola $H(k, \ell) = 0$, and α_t is the derivative of k with respect to ℓ of the hyperbola at the point $(k, \ell) = (\beta_t, \beta_t)$. Explicitly,

$$\beta_t = \frac{3}{2}t + 1 + \sqrt{\frac{9}{4}t^2 + 5t} \quad \text{and} \quad \alpha_t = \frac{t + 1 - \beta_t}{\beta_t - 2t - 1}.$$

To solve the primal-dual pair (4)-(5), notice that $(\bar{k}, \bar{\ell}) = (\beta_t, \beta_t)$ and $(\bar{x}, \bar{y}) = (\frac{1+\alpha_t}{\alpha_t - 1}, \frac{2}{1-\alpha_t})$ are feasible in (4) and (5), respectively, both with the objective value $2\beta_t$. Thus, it follows by weak duality that $(\bar{k}, \bar{\ell})$ is optimal in the LP (4). Since $(\bar{k}, \bar{\ell})$ is also feasible in the NLP (2), it is optimal there too.

We will also solve the auxiliary NLP (3), just as we solved (2). First, we find an LP relaxation of NLP (3), the LP (6), and write its dual, the LP (7),

$$\begin{array}{ll} \min & [1\ 1]\,[k\ \ell]^T \\ \text{s.t.} & \begin{bmatrix} 1 & -1 \\ 1 & -\delta_t \end{bmatrix} \begin{bmatrix} k \\ \ell \end{bmatrix} \geq \begin{bmatrix} 1 \\ \gamma_t - \delta_t\gamma_t + 1 \end{bmatrix} \\ \text{where} & k, \ell \geq 0 \end{array} \quad (6)$$

$$\begin{array}{ll} \max & [1\ (\gamma_t - \delta_t\gamma_t + 1)]\,[x\ y]^T \\ \text{s.t.} & \begin{bmatrix} 1 & 1 \\ -1 & -\delta_t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \text{where} & x, y \geq 0 \end{array} \quad (7)$$

where γ_t and δ_t are defined analogously to α_t and β_t. The number γ_t is such that $(\gamma_t + 1, \gamma_t)$ is the point of intersection between the line $k = \ell + 1$ and the hyperbola $H(k, \ell) = 0$, and δ_t is the derivative of k with respect to ℓ of the hyperbola at the point $(k, \ell) = (\gamma_t + 1, \gamma_t)$. Explicitly,

$$\gamma_t = \frac{3}{2}t + \frac{1}{2} + \sqrt{\frac{9}{4}t^2 + \frac{11}{2}t + \frac{1}{4}} \quad \text{and} \quad \delta_t = \frac{t - \gamma_t}{\gamma_t - 2t - 1}.$$

Observe that $(\bar{k}, \bar{\ell}) = (\gamma_t + 1, \gamma_t)$ and $(\bar{x}, \bar{y}) = (\frac{1+\delta_t}{\delta_t - 1}, \frac{2}{1-\delta_t})$ are feasible in (6) and (7), respectively, both with objective value $2\gamma_t + 1$. Thus, it follows by weak duality that $(\bar{k}, \bar{\ell})$ is optimal in (6). Since $(\bar{k}, \bar{\ell})$ is feasible also in the NLP (3), it is optimal there too.

We are now ready to solve the original integer NLP (1). In the $t = 1$ case, the optimal value of (1) is at least the ceiling of the minimum of the optimal values of (4) and (6), which is 11. Since $(k, \ell) = (\gamma_1 + 1, \gamma_1) = (6, 5)$ realizes this bound, it is optimal in (1). For the $t = 2$ case, $\gamma_2 = 8$ is an integer. Hence, $(k, \ell) = (9, 8)$ is an optimal integer solution to (3). Because the only integer points feasible in (2) but not in (3) are on the line $k = \ell$, the least of which is $(k, \ell) = (\lceil\beta_2\rceil, \lceil\beta_2\rceil) = (9, 9)$, we have that $(k, \ell) = (8, 9)$ is optimal in (1). For $t > 2$, we observe $3t + 3 > \beta_t, \gamma_t > 3t + 2$. Since $(k, \ell) = (\gamma_t + 1, \gamma_t)$ is optimal in (3), all feasible integers $k > \ell$ have $k + \ell \geq \lceil 2\gamma_t + 1\rceil \geq 6t + 6$. Since the least feasible integer $k = \ell$ is $\lceil\beta_t\rceil = 3t + 3$, we have that $(k, \ell) = (3t + 3, 3t + 3)$ is optimal in (1). The result follows from these solutions to (1) and Bound (1) of Lemma 2. □

We remark that one could alternatively perform the above proof by finding points satisfying the Karush-Kuhn-Tucker conditions in NLPs (2) and (3).

3.2 Upper Bound

To bound the minimum area of a t-TCP from above by $n \in \mathbb{N}$ inclusive, it suffices to find a t-TCP of area n. For $t = 1, 2$, the examples given in Figures 2(a,b) are enough. For $t > 2$, we require a general construction.

Lemma 4 *For $t \geq 3$, $m_t \leq 6(t + 1)^2 - 1$.*

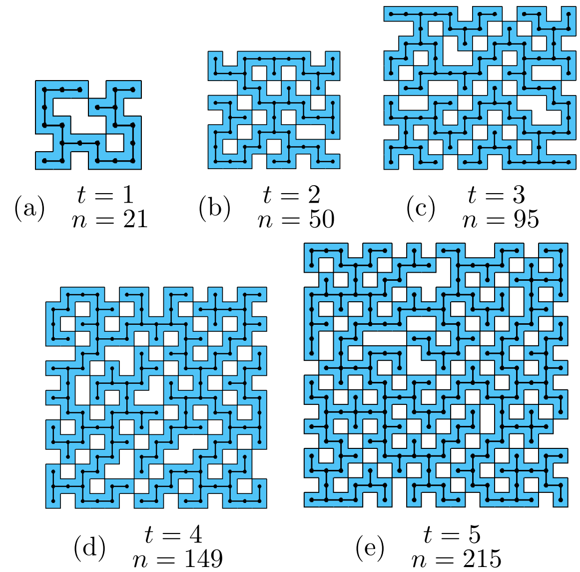


Figure 2: Minimum-area TCPs for $1 \leq t \leq 5$
.

Proof. Consider the following construction, in four steps. It is illustrated in Figure 3 in the $t = 3$ case.

1. Create a collection of cells, placing one cell about a point $(x, y) \in \{0, 1, 2, \cdots, 3t+2\} \times \{0, 1, 2, \cdots, 3t+2\}$ if and only if the sum $(x + y)$ is not congruent to 2 modulo 3. This collection has $6(t + 1)^2$ cells, and $2(t + 1)$ connected components. Each row and column has at least t gaps, and some have more.

2. There are $(t+1)$ columns (resp., rows) with $(t+1)$ gaps, at $x \equiv 2 \mod 3$ (resp., $y \equiv 2 \mod 3$). Place more cells about points of the form $(3i+1, 3i+1)$, for $0 \leq i \leq t$. The resulting collection of cells has $6(t + 1)^2 + (t + 1)$ cells, and $(t + 1)$ connected components. Each row and column retains at least t gaps.

3. Step 2 created multiply connected components. Therefore, we may remove the $2t + 2$ cells about points of the form $(3i + 1 \pm 1, 3i + 1 \pm 1)$ for $0 \leq i \leq t$ without creating more connected components. There are still at least t gaps in each column and row, since $(3i, 3i)$ is adjacent to $(3i+1, 3i)$ and $(3i, 3i+1)$, while $(3i+2, 3i+2)$ is adjacent to $(3i+2, 3i+1)$ and $(3i+1, 3i+2)$. The result is a collection of $6(t + 1)^2 - (t + 1)$ cells.

4. We can connect the remaining $(t + 1)$ connected components to each other with t additional cells, centered about points of the form $(3i, 3i - 1)$ for $1 \leq i \leq t$. No such addition breaks t-fold concavity, since $(3i, 3i-1)$ is not adjacent to either $(3i+1, 3i+1)$ or $(3i, 3i)$, each removed in Step 3. We are left with a t-TCP of area $6(t + 1)^2 - 1$.

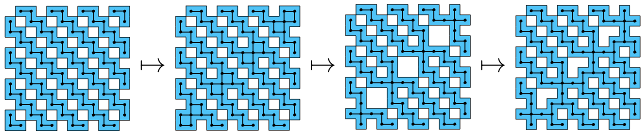


Figure 3: Constructing a t-TCP of area $6(t+1)^2-1$.

Since this construction works for all $t > 2$, the claim is proved. □

Note that the examples produced by the above construction are different from the ones given in Figure 2.

Corollary 5 *For $t \geq 1$, there are at least 2^t-many t-TCPs of area $6(t+1)^2-1$.*

Proof. In Step 4, we could have just as well placed a cell about the point $(3t, 3t+1)$ whenever we placed one about $(3t+1, 3t)$, giving t binary choices. □

3.3 Structure of Minimum-Area TCPs

Figure 2 shows some minimum-area t-TCPs for $1 \leq t \leq 5$. Lemma 6 below characterizes the bounding boxes of minimal t-TCPs. The existence of non-square minimal t-TCP for $t > 3$ is an open question. The construction in the proof of Theorem 4 shows that there exists a minimal t-TCP in a $(3t+3, 3t+3)$ bounding box for all $t > 2$. However, the 3-TCP with a (13,11) bounding box shown in Figure 2(c) is currently the only example of a minimal t-TCP that does not have a square bounding box for $t > 2$. The following results also relate the dimension of a t-TCPs bounding box to its connectivity and its concavity.

Lemma 6 *If a t-TC polyomino has a (k, ℓ) bounding box and its area is $(t+1)(k+\ell)-1$, then*
(i) it is a tree; and
(ii) it has exactly t gaps in every row and every column.

Proof. We prove the contrapositive. If P is an area-n t-TCP that is not a tree, the bound for the number of hidden edges (see the proof of Lemma 2) becomes $h \geq 2n$. If P has more than t gaps in some row or column, the bound on the number of inside edges becomes $i \geq 2t(k+\ell)+2$. In either case, we get $n \geq (t+1)(k+\ell)$ given that $i+o+h = 4n$, hence $n \neq (t+1)(k+\ell)-1$. The claim follows. □

We now present our second main result.

Theorem 7 *Suppose that P is a minimum-area t-TCP of area n whose bounding box is $B = (k, \ell)$ (for $k \geq \ell$). If $t = 1$, then $B = (6, 5)$. If $t = 2$, then $B = (9, 8)$. Otherwise, if $t \geq 3$, then B is either $(3t+3, 3t+3)$ or $(3t+4, 3t+2)$. Moreover, P is a tree and it has exactly t gaps in every row and every column.*

Proof. Let P be a t-TCP with area m_t in a (k, ℓ) bounding box. We claim that

$$(t+1)(k+\ell)-1 = m_t. \tag{8}$$

By Lemma 2, we have that $m_t \geq (t+1)(k+\ell)-1$. The pair (k, ℓ) is feasible in the integer NLP (1) because P is a t-TCP. Since m_t is equal to the lower bound given by Lemma 3, m_t is the minimum of $(t+1)(k+\ell)-1$ for feasible (k, ℓ) pairs, that is, $m_t \leq (t+1)(k+\ell)-1$. Hence, Equation (8) holds, and by Lemma 6 we have that all minimal t-TCPs are trees and have exactly t gaps everywhere.

It is easy to check that the unique solutions that satisfy Equation (8) and the relations in Lemma 2 are $(k, \ell) = (6, 5)$ in the $t = 1$ case and $(k, \ell) = (9, 8)$ in the $t = 2$ case. A manual inspection of all 1-TCPs (provided in reference [2]) is also available for $t = 1$.

For $t \geq 3$, notice that the solutions to Equation (8) with $k \geq \ell$ take the form $(k, \ell) = (3t+3+\Delta, 3t+3-\Delta)$ for some non-negative integer $\Delta \geq 0$. Expanding and rearranging the relations in Lemma 2 in this case give $t(1-\Delta)+4-\Delta^2 \geq 0$, which is possible only if $\Delta = 0$ or 1. □

4 Equality of the t-TCP Growth Constants to λ

It is straightforward to prove the existence of the t-TCP growth constants via a concatenation argument and supermultiplicativity, as was done previously for 1-TCPs [2] and is common for other families of lattice animals [1, 5, 10].

Theorem 8 *For all $t > 0$, $\lambda_{\kappa_t} := \lim_{n\to\infty} \sqrt[n]{\kappa_t(n)}$ exists. Moreover, $\lambda_{\kappa_t} := \sup_n \sqrt[n]{\kappa_t(n)}$.*

Proof. Since every t-TCP is a polyomino, $\sqrt[n]{\kappa_t(n)} \leq \sqrt[n]{A(n)}$. Additionally, $\sqrt[n]{A(n)} \to \lambda$ as $n \to \infty$. We conclude that the sequence $\{\sqrt[n]{\kappa_t(n)}\}_{n>0}$ is bounded. We will also show that for all $t, n, m > 0$, $\kappa_t(n) \cdot \kappa_t(m) \leq \kappa_t(n+m)$, *i.e.*, $\{\kappa_t(n)\}$ is supermultiplicative. The result then follows from Lemma 1 of Ref. [10] which states precisely the existence of the hypothesized limits for supermultiplicative sequences that are bounded as above.

To see the supermultiplicative relation, we concatenate two t-TCPs of sizes n and m. Given $P_1 \in \kappa_{t,n}$ and $P_2 \in \kappa_{t,m}$, let P_3 be the union of P_1 and the translation of P_2 such that its lexicographically smallest cell lies immediately to the right of P_1's greatest cell. Then, P_3 uniquely determines an element of $\kappa_{t,n+m}$ since all rows and columns still have at least t gaps, and the original pair P_1, P_2 can be determined uniquely from P_3 by separating the n lexicographically-smallest from the m lexicographically-biggest cells of P_3.

To prove $\lambda_{\kappa_t} := \sup_n \sqrt[n]{\kappa_t(n)}$, observe that for each n, the supermultiplicative relation implies that the subsequence $\{ \sqrt[n2^m]{\kappa_t(n2^m)} : m \geq 0\}$ is increasing. Indeed, its limit must be λ_{κ_t}, so $\sqrt[n]{\kappa_t(n)} \leq \lambda_{\kappa_t}$ for all $n > 0$ $\square$

Theorem 8 can be used for obtaining lower bounds on λ_{κ_t}. If $\kappa_t(n) \geq x$, then $\lambda_{\kappa_t} \geq \sqrt[n]{x}$. That is how the best known lower bound on λ_{κ_1}, which is 2.4474, was found (see reference [2]). However, the following construction does better. Compare the previous known bound to the one given in Corollary 10 below.

Theorem 9 *The growth constant for t-TCPs, λ_{κ_t}, equals λ for all $t > 0$.*

Proof. We partition the set of all n-ominoes into a polynomial number of subsets. Given any polyomino P, we define the following quantities.

$$\begin{aligned} X_{span}(P) &= \max\{x : (x,y) \in P \text{ for some } y\}, \\ Y_{span}(P) &= \max\{y : (x,y) \in P \text{ for some } x\}, \\ X_0^-(P) &= \min\{x : (x,0) \in P\}, \\ X_0^+(P) &= \min\{x : (x, Y_{span}(P)) \in P\}, \\ Y_0^-(P) &= \min\{y : (0,y) \in P\}, \\ Y_0^+(P) &= \min\{y : (X_{span}(P), y) \in P\}. \end{aligned}$$

Note that if the span of a polyomino P in either of the axes is d, then the coordinates of cells of P along that axis are in the range $[0, d-1]$. Then, the set $P_n[a,b,c,d,e,f]$ is defined as

$$\begin{aligned} P_n[a,b,c,d,e,f] = \{P \in A_n :\ & X_{span}(P) = a, \\ & Y_{span}(P) = b, X_0^-(P) = c, X_0^+(P) = d, \\ & Y_0^-(P) = e, Y_0^+(P) = f\}. \end{aligned}$$

See Figure 4 for an illustration of a typical member of $P_n[a,b,c,d,e,f]$. It follows from the connectedness of P that $P_n[a,b,c,d,e,f] = \emptyset$ if any of a,b,c,d,e,f are greater than n, hence,

$$A_n = \bigcup_{\substack{0<a,b\leq n \\ 0\leq c,d,e,f<n}} P_n[a,b,c,d,e,f]. \quad (9)$$

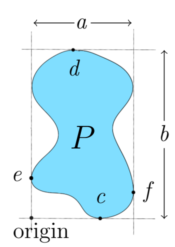


Figure 4: A typical member of $P_n[a,b,c,d,e,f]$.

Therefore, there exist $a^\circ, b^\circ, c^\circ, d^\circ, e^\circ, f^\circ$ for which $|P_n[a^\circ, b^\circ, c^\circ, d^\circ, e^\circ, f^\circ]| \geq \left(\frac{A(n)}{n^6}\right)$. For each $i,j \in \{0,1,\ldots,t\}$, let $\pi_{i,j}$ be any element of $P_n[a^\circ, b^\circ, c^\circ, d^\circ, e^\circ, f^\circ]$. There are at least $\left(\frac{A(n)}{n^6}\right)^{(t+1)^2}$ choices for $\{\pi_{i,j}\}_{0\leq i,j\leq t}$.

We now construct to each choice of $\{\pi_{i,j}\}$ a unique t-TCP. First we define the polyomino B ( ), and its 90° clockwise-rotated version, $B^{\circlearrowright}$. We use the notation P_G as defined in the introduction. We set $B := P_G$, where $G := \{(0,0),(1,0),(1,1),(2,1),(3,1),(3,0),(4,0)\}$, and $B^{\circlearrowright} := P_{G^{\circlearrowright}}$, where $G^{\circlearrowright} := \{(0,0),(0,1),(1,1),(1,2),(1,3),(0,3),(0,4)\}$.

Let $P + \vec{v}$ denote the translation of a polyomino P by a vector $\vec{v} \in \mathbb{Z}^2$. We are now ready to define our constructed t-TC polyomino, $\mathrm{TCP}(\{\pi_{i,j}\}_{0\leq i,j\leq t})$. Define $\{\pi'_{i,j}\}_{0\leq i,j\leq t}$, $\{\hat{\pi}_{i,j}\}_{0\leq i,j\leq t}$, $\{B_{i,j}\}_{0\leq i<t,\,0\leq j\leq t}$, and $\{B^{\circlearrowright}_{i,j}\}_{0\leq i\leq t,\,0\leq j<t}$, by the following rules.

$$\pi'_{i,j} = \begin{cases} \pi_{i,j} & i,j \text{ even} \\ \text{reflection of } \pi_{i,j} \text{ through the line } y = b^\circ/2 & i \text{ even}, j \text{ odd} \\ \text{reflection of } \pi_{i,j} \text{ through the line } x = a^\circ/2 & i \text{ odd}, j \text{ even} \\ \text{reflection of } \pi_{i,j} \text{ through } y = \frac{b^\circ}{2} \text{ and } x = \frac{a^\circ}{2} & i,j \text{ odd} \end{cases}$$

$$\hat{\pi}_{i,j} = \pi'_{i,j} + (i(a^\circ+5), j(b^\circ+5))$$

$$B_{i,j} = \begin{cases} B + (i(a^\circ+5)+a^\circ,\ j(b^\circ+5)+f^\circ) & i,j \text{ even} \\ B + (i(a^\circ+5)+a^\circ,\ j(b^\circ+5)+b^\circ-f^\circ-1) & i \text{ even}, j \text{ odd} \\ B + (i(a^\circ+5)+a^\circ,\ j(b^\circ+5)+e^\circ) & i \text{ odd}, j \text{ even} \\ B + (i(a^\circ+5)+a^\circ,\ j(b^\circ+5)+b^\circ-e^\circ-1) & i,j \text{ odd} \end{cases}$$

$$B^{\circlearrowright}_{i,j} = \begin{cases} B^{\circlearrowright} + (i(a^\circ+5)+d^\circ,\ j(b^\circ+5)+b^\circ) & i,j \text{ even} \\ B^{\circlearrowright} + (i(a^\circ+5)+c^\circ,\ j(b^\circ+5)+b^\circ) & i \text{ even}, j \text{ odd} \\ B^{\circlearrowright} + (i(a^\circ+5)+a^\circ-d^\circ-1,\ j(b^\circ+5)+b^\circ) & i \text{ odd}, j \text{ even} \\ B^{\circlearrowright} + (i(a^\circ+5)+a^\circ-c^\circ-1,\ j(b^\circ+5)+b^\circ) & i,j \text{ odd} \end{cases}$$

Thus, $B_{i,j}$ intersects each of $\hat{\pi}_{i,j}$ and $\hat{\pi}_{i+1,j}$ in one edge, and $B^{\circlearrowright}_{i,j}$ intersects each of $\hat{\pi}_{i,j}$ and $\hat{\pi}_{i,j+1}$ in one edge. Finally, define

$$TCP(\{\pi_{i,j}\}_{0\leq i,j\leq t}) = \left(\bigcup_{0\leq i,j\leq t} \hat{\pi}_{i,j}\right) \cup \left(\bigcup_{i=0}^{t-1}\bigcup_{j=0}^{t} B_{i,j}\right) \cup \left(\bigcup_{i=0}^{t}\bigcup_{j=0}^{t-1} B^{\circlearrowright}_{i,j}\right).$$

See Figure 5 for an illustration of this construction. It is evident that $\mathrm{TCP}(\{\pi_{i,j}\})$ uniquely determines $\{\pi_{i,j}\}$. It is easily verified by inspecting rows and columns that $\mathrm{TCP}(\{\pi_{i,j}\})$ is indeed t-TC, although we must be careful with the following special case. If e° or f° is $b^\circ-1$ or 0, then we must redefine $B_{i,t}$ for $0 \leq i < t$, replacing B ( ) in the above with its vertical reflection ( ) in the line $y = \frac{1}{2}$. Without this hack, the top row of $\mathrm{TCP}(\{\pi_{i,j}\})$ may not be t-TC, consisting only of the $B_{i,t}$ kinks. The same applies to $B^{\circlearrowright}_{t,j}$ for the case in which c° or d° is $a^\circ-1$ or 0, where we must replace $B^{\circlearrowright}$ with its horizontal reflection in the line $x = \frac{1}{2}$.

The constructed t-TC polyomino $\mathrm{TCP}(\{\pi_{i,j}\})$ has $\varphi := (t+1)^2 n + 14t(t+1)$ cells. The $14t(t+1)$ term comes from the 7 cells in each of $B_{i,j}$ and $B^{\circlearrowright}_{i,j}$. Therefore, we have that $\kappa_t(\varphi) \geq \left(\frac{A(n)}{n^6}\right)^{(t+1)^2}$, and hence we have that $\sqrt[\varphi]{\kappa_t(\varphi)} \geq \sqrt[\varphi]{\left(\frac{A(n)}{n^6}\right)^{(t+1)^2}}$.

We now let $n \to \infty$. Note that the indices φ define a subsequence of the sequence $\kappa_t(n)$. Since the sequence $\sqrt[n]{\kappa_t(n)}$ converges, so does the subsequence, and to the same limit. The right side of the final inequality shows

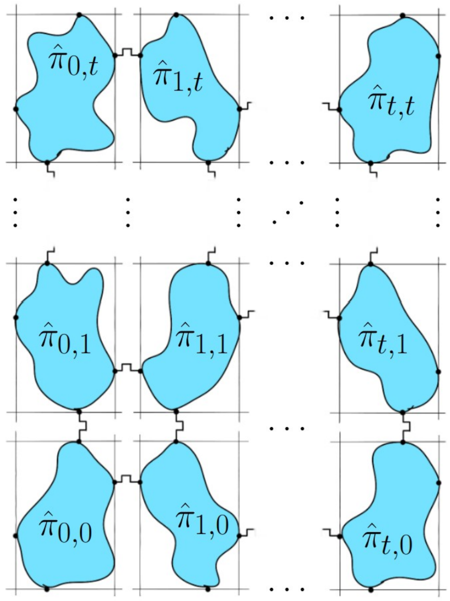


Figure 5: A "blob" representation of the polyomino $\mathrm{TCP}(\{\hat{\pi}_{i,j}\}_{0\le i,j\le t})$. The blobs represent $\hat{\pi}_{i,j}$s, and the "squiggles" between them the $B_{i,j}$s and $B^{\circlearrowleft}_{i,j}$s. Notice how the $\hat{\pi}_{i,j}$s are flipped throughout the construction to match along their boundaries.

that the limit is at least λ. However, the limit cannot exceed λ, because t-TCPs are a proper subset of all polyominoes. Therefore, it must be that $\lambda_{\kappa_t} = \lambda$. □

Corollary 10 *For all $t > 0$, we have that $4.0025 \le \lambda_{\kappa_t} \le 4.5252$.*

Proof. These are just the best known lower [3] and upper [4] bounds on λ. □

We have just proved that t-TCPs are not exponentially rare in the polyominoes. That begs the question; how common are t-TCPs? The next theorems show that they are not overwhelmingly so: The proportion of polyominoes that are t-TCPs is bounded away from 1.

First, we need a ratio-limit theorem.

Theorem 11 *(Ratio Limit Theorem for t-TCPs) $\lim_{n\to\infty} \frac{\kappa_t(n+1)}{\kappa_t(n)}$ exists and equals λ_{κ_t}.* □

The proof is given in the full version of the paper.

Theorem 12 $\limsup_{n\to\infty} \frac{\kappa_t(n)}{A(n)} \le \frac{\lambda}{\lambda+4(t+1)}$ $\forall t > 0$.

Proof. We map every t-TCP of size $n-1$ to all elements of A_n obtained by attaching one cell immediately to the right (resp., left) of any cell in its rightmost (resp., leftmost) column, or above (resp., below) any cell in its topmost (resp., bottommost) row. This way, every polyomino in $\kappa_{t,n-1}$ is mapped to at least $4(t+1)$ polyominoes in A_n, all images are distinct, and all images are not t-TCPs. Hence, we have that $A(n) \ge \kappa_t(n) + 4(t+1)\kappa_t(n-1)$, that is, $A(n)/\kappa_t(n) \ge 1 + 4(t+1)\kappa_t(n-1)/\kappa_t(n)$. Therefore, by the Ratio Limit Theorem for t-TCPs (Theorem 11),

$$\limsup \frac{\kappa_t(n)}{A(n)} \le \lim_{n\to\infty} \frac{1}{1+\frac{4(t+1)\kappa_t(n-1)}{\kappa_t(n)}} = \frac{1}{1+\frac{4(t+1)}{\lambda}} = \frac{\lambda}{\lambda+4(t+1)}.$$

□

Note that this does *not* imply that $\kappa_t(n)/A(n)$ (t fixed) converges to a positive value (as a function of n), or converges at all, when $n \to \infty$. For example, if $A(n) \sim c_0 n^{\theta_0}\lambda^n$ (which is widely believed) and $\kappa_t(n) \sim c_t n^{\theta_t}\lambda^n$, where $\theta_t \le \theta_0$, then $\lim_{n\to\infty} \kappa_t(n)/A(n)$ would be 0 if $\theta_t < \theta_0$ but nonzero if $\theta_t = \theta_0$. However, since $\lim_{t\to\infty} \lambda/(\lambda+4(t+1)) = 0$, we conclude that the limiting fraction of t-TCPs out of all polyominoes vanishes as t tends to ∞.

5 Conclusion

In this paper, we make a natural generalization of TCPs to t-TCPs, generalized previous results on TCPs to t-TCPs, and proved new results on TCPs in the more general t-TCP case. To answer the minimum-area problem for t-TCPs, we find that m_t grows quadratically with t (Theorem 1).

We also prove that for a fixed t, t-TCPs are not exponentially rare in the regular (those not necessarily TC) polyominoes (Theorem 9). This begs the question, exactly how common are t-TCPs in the regular polyominoes? From the bound $\kappa_t(\varphi) \ge (A(n)/n^6)^{(t+1)^2}$ in the proof of Theorem 9, we find that the widely believed relation $c_0 n^{\theta_0}\lambda^n \le A(n)$ (for some constants c_0, θ_0), together with the known relation $A(n) \le \lambda^n$, would imply that $\kappa_t(n)/A(n) = \Omega\left(n^{(\theta_0-6)(t+1)^2}\right)$. We also have an asymptotic upper bound on $\kappa_t(n)/A(n)$ in Theorem 12. Our question in this regard is: For a fixed t, do t-TCPs form an asymptotically positive fraction of all polyominoes? Looking at the known values of $\kappa_t(n)/A(n)$, it seems plausible that it is indeed a positive fraction, but the data are insufficient to make confident claims. In the case that rigorous proof eludes, we may still gain insight by measuring the prevalence of t-TCPs in Monte Carlo samples of polyominoes, along the lines of reference [9].

We also notice that the definitions of *minimality* (not minimum area), *primitivity*, and *saturation* given in reference [2] have clear t-fold generalizations, and questions 3–6 therein can be posed in the t-fold case too.

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