Guarding Polygons With Mutually Visible π-Guards*

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Abstract

We study a version of the polygon guarding problem in which we want to guard the polygon with a minimum number of vertex guards with 180° field of view such that for each guard $\mathbf{g}$ there is a guard $\mathbf{g}'$ where $\mathbf{g}$ and $\mathbf{g}'$ are mutually visible. Let $g(n)$ be the minimum number of such guards over all polygons of size n. We show that $\frac{2n-2}{3} \leqslant g(n) \leqslant \frac{4n}{5}$. We define $\bar{g}(n)$ analogously for orthogonal polygons, and show that $\frac{3n-4}{7} \leqslant \bar{g}(n) \leqslant \frac{n}{2}$. The lower bounds are existential in the sense that there are polygons that need these many guards.

1 Introduction

The classic art gallery problem, posed by Victor Klee in 1973 [8], asks for "*the minimum number of guards required to monitor every point within an art gallery with n walls*". In this problem, each guard is considered to be a point g in the polygon (including boundary) that can see any point p, where the segment gp lies inside the polygon. In 1975, Chvátal [5] showed that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and sometimes necessary to cover a polygon with n vertices. Chvátal's proof is by induction on a triangulation graph of the polygon. In 1978, Fisk [6] provided a shorter proof using a 3-coloring of the triangulation graph. The initial appearance of other variations of the art gallery problem followed these foundational results, including scenarios with mobile guards [14], guards with limited visibility [3, 18, 19] or mobility [1, 4], and guarding orthogonal polygons [9, 13]. For more related results, we refer the interested readers to the book by O'rourke [15].

One of the variations of the art gallery problem is the concept of *cooperative guards* [10], which is also known as *connected guards* [17]. The goal of this problem is to find the minimum number of guards such that their visibility graph is connected. In another variation [11, 17, 20], known as *guarded guards* (*weakly cooperative guards*, or *watched guards*), the goal is to find the minimum number of guards such that each guard is visible from some other guard. For example, in Figure 1(a) three guards are sufficient for guarding the polygon, while in Figure 1(b), the guards must guard each other mutually, resulting in a total number of 4 guards, each of which is being guarded by at least one other guard. The main motivation of these variations is that if something goes wrong with one guard, at least one other guard can be notified.

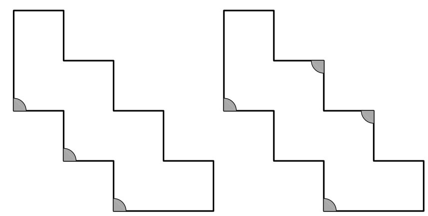

Figure 1: A difference between (a) classic guards, and (b) guarded guards.

The guarded guards problem was introduced by Liaw et. al. [11] in 1994, where they presented an optimal linear-time algorithm for 1-spiral polygons.[1] In 1995, Hernandez [16] proved an upper bound of $\lfloor \frac{n}{3} \rfloor$ for orthogonal polygons. In 2002, Zylinski [20, 21] provided tight bounds of $\lfloor \frac{2n}{5} \rfloor$ for monotone and spiral polygons, and $\lfloor \frac{3n-1}{7} \rfloor$ for star-shaped polygons. In the same year, independently, Michael and Pinciu [17] showed that $\lfloor \frac{3n-1}{7} \rfloor$ guards for simple polygons are always sufficient and sometimes necessary. They also provided a shorter proof on the upper bound of orthogonal polygons. Regarding computational complexity, Liaw et. al. proved that the guarded guards problem is $\mathcal{NP}$-hard [11].

In the above problems the guards are assumed to have 360° field of view. In recent years, there has been an increased interest in guards with bounded field of view that only see an angle $\alpha < 360°$ [3, 18, 19]. For 360° field of view, if a guard g_1 sees g_2, then g_2 also sees g_1, which means that they are *mutually visible*. However, this property does not hold for $\alpha < 360°$. Florentino et al. [7] studied a version of the connected guard problem with $\alpha = 180°$, where the mutual visibility graph must be connected.

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[1] A polygon is 1-spiral if its boundary has exactly one chain of reflex vertices.

1.1 Preliminaries

A *triangulation* of a polygon P refers to the division of P into a set of triangles, which are non-overlapping and cover the entire area of the polygon. The corners of triangles are at vertices of P. The *dual graph* of a triangulation is constructed by defining a vertex for each triangle in the triangulation; edges are drawn between vertices if the corresponding triangles share a common edge [15]. The dual of triangulation is a tree with vertices of degree ≤ 3.

An *orthogonal polygon* is defined as a polygon whose edges meet at right angles, resulting in edges that are either horizontal or vertical. An orthogonal polygon admits a *convex quadrangulation* which is a partitioning of the polygon into convex quadrilaterals [9]. The *dual of convex quadrangulation* is derived analogously to the dual of triangulation. The dual of quadrangulation is a tree with exactly $\frac{n-2}{2}$ vertices of degree $\leqslant 4$ [15].

1.2 Our Contributions

We study the guarded guards problem with guards of 180° field of view. In this problem, we are given a simple polygon P with n vertices. The goal is to place the minimum number of guards with 180° field of view at vertices of P to cover the entire polygon P, such that for each guard g, there is a guard g' such that g and g' see each other, in other words, g and g' are mutually visible. In this problem:

1. The guards must be placed at the vertices.
2. Each guard has 180° field of view.
3. Each guard must be mutually visible by some other guard (Figure 2).
4. The guards cannot be placed outwards (i.e. towards the exterior of the polygon as in Figure 11).
5. If two guards are placed at the same vertex, then they cannot be mutually visible to each other.

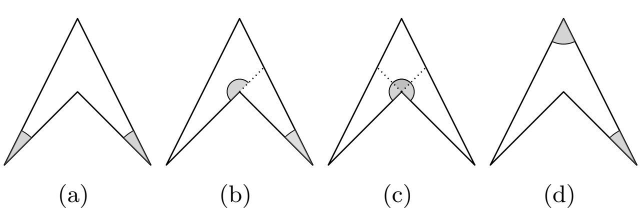


Figure 2: In (a), (b), (c) the guards cover the polygon but do not see each other mutually. (d) is a valid guarded guarding.

The assumption that, two guards placed at the same vertex are not considered visible to each other, is to avoid collusion among the guards.

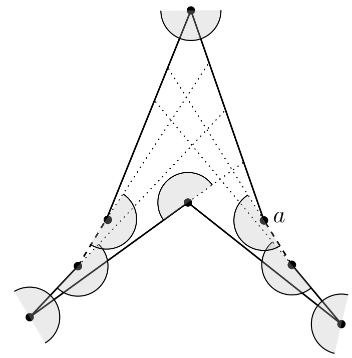


Figure 3: The necessity of two guards on some vertex.

Table 1: Bounds for different variants of the problem.

Polygon	Field of View	Bound
Simple	360°	$\lfloor \frac{3n-1}{7} \rfloor$ [12, 20]
Orthogonal	360°	$\lfloor \frac{n}{3} \rfloor$ [12, 16]
Simple	180°	$[\frac{2n-2}{3}, \frac{4n}{5}]$ (Thm. 2)
Orthogonal	180°	$[\frac{3n}{7}, \frac{n}{2}]$ (Thm. 9)

We define $g(n)$ to be the minimum number such that any polygon with n vertices can be guarded by at most $g(n)$ such guards. We define $\bar{g}(n)$ analogously for orthogonal polygons, where the edges are axis-aligned.

Placement of two guards at one vertex is sometimes necessary. Figure 3 shows an arbitrary large polygon, such that if we place one guard at each vertex there is still a guard that is not mutually visible by any other guard (vertex a in the figure).

In Section 2 we study simple polygons and prove that $\frac{2n-2}{3} \leqslant g(n) \leqslant \frac{4n}{5}$. In Section 3 we study the orthogonal polygons and prove that $\frac{3n-4}{7} \leqslant \bar{g}(n) \leqslant \frac{n}{2}$. Table 1 summarizes existing bounds for angle 360° and our bounds for angle 180°. In the rest of the paper the term "guard" refers to a guard of 180° field of view.

2 Simple Polygons

In this Section we prove that any n-vertex polygon can be mutually guarded by at most $\frac{4n}{5}$ guards. Moreover, there are n-vertex polygons that require $\frac{2n-2}{3}$ guards. Therefore, $\frac{2n-2}{3} \leqslant g(n) \leqslant \frac{4n}{5}$. In section 2.1 we present a lower bound example, and in section 2.2 we prove the upper bound. Our proof is by induction. It cuts a small portion of the polygon, guard it with a few mutually visible guards, and apply induction on the rest of the polygon. This is a standard technique used in many visibility guarding problems [14, 17].

2.1 The Lower Bound

For every $m \geqslant 1$ we construct a polygon P with $n = 3m+1$ vertices that requires $2m$ guards, which implies the lower bound. Our polygon is illustrated in Figure 4. For any $i \leqslant m$, we identify a part in P that consists of vertices a_i, b_i, c_i, and a_{i+1}. In each part i, we consider two points p_i and q_i such that they are only visible from the vertices of its corresponding part. Placing one guard g_i on either one of a_i, b_i, or a_{i+1} is not sufficient for guarding both p_i and q_i; if g_i is placed on c_i, then it covers part i, but it is not covered by any other guard. Thus, we need two guards in part i to mutually guard p_i and q_i. Since there are m such parts in P, and each part requires two guards, polygon P requires a total amount of $2m = \frac{2(n-1)}{3}$ guards. Therefore, $g(n) \geqslant \frac{2n-2}{3}$.

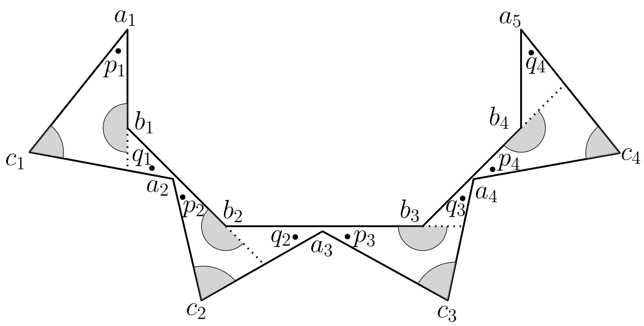


Figure 4: A polygon that requires at least $\frac{2n-2}{3}$ guards.

2.2 The Upper Bound

We prove that any n-vertex simple polygon P can be mutually guarded by at most $\frac{4n}{5}$ guards. Hence, $g(n) \leqslant \frac{4n}{5}$. Our proof is by induction on the number of vertices of P. We cut a small part of P and guard it, and recur on the remaining part of the polygon. We continue the induction until the remaining polygon is one of our base cases ($n \leqslant 7$).

The following lemma is implied from a result of [2]:

Lemma 1 *Let P be a polygon with $n \geqslant 8$ vertices. There exists a diagonal d that divides P into two polygons P' and P'' with m and $n-m+2$ vertices, respectively, for some $m \in \{5,6,7\}$.*

Observation 1 *Let P be a polygon with $n \geqslant 8$ vertices. If P can be partitioned into two polygons P' and P'' by a diagonal, with $n' \geqslant 3$ and $n'' \geqslant 3$ vertices, respectively, then $g(n) \leqslant g(n') + g(n'')$.*

Theorem 2 *For every $n \geqslant 3$:*

$$g(n) = \begin{cases} 2 & \text{if } n = 3,4,5 \\ 3 & \text{if } n = 6 \\ 4 & \text{if } n = 7 \\ \frac{4n}{5} & \text{if } n \geqslant 8. \end{cases}$$

Proof. The proof is by induction on n. The cases were $3 \leqslant n \leqslant 7$ serve as base cases, and are proved later in Section 2.2.1.

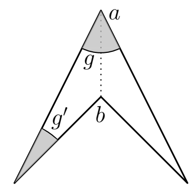


Figure 5: Guarding of P_4.

We proceed the induction step for $n \geqslant 8$. By Lemma 1, there exists a diagonal that partitions P into two polygons P' and P'' of sizes m and $n-m+2$, respectively, such that $m \in \{5,6,7\}$. By Observation 1, $g(n) \leqslant g(n-m+2) + g(m)$. Substituting the possible values of m into the formula confirms the induction step, as shown below:

$$g(n) \leq g(n-3) + g(5) \leq \frac{4(n-3)}{5} + 2 \leq \frac{4n}{5},$$
$$g(n) \leq g(n-4) + g(6) \leq \frac{4(n-4)}{5} + 3 \leq \frac{4n}{5},$$
$$g(n) \leq g(n-5) + g(7) \leq \frac{4(n-5)}{5} + 4 \leq \frac{4n}{5}.$$

□

2.2.1 Base Cases

The base cases happen when $n \leqslant 8$. For $n = 3$, we need exactly two mutual guards to guard the entire polygon. As every polygon with $n = 3$ vertices is convex, one guard is sufficient to guard the entire area of the polygon. We require another guard to guard the first guard, resulting in a total of 2 guards. In Lemmas 3, 4, 5, and 6 we show the base cases where $n = 4$, $n = 5$, $n = 6$, and $n = 7$, respectively. Note that in the proofs, the term P_n presents a polygon with n vertices.

Since the sum of a quadrilateral's internal angles is 360°, the following observation holds:

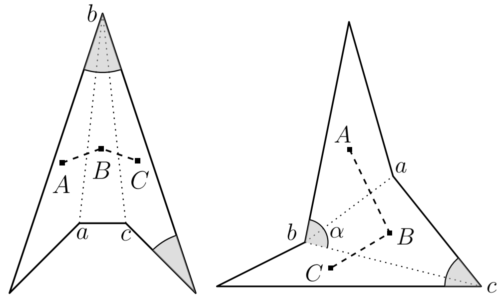
Observation 2 *Let Q be a quadrilateral. Among each pair of opposite angles in Q, at least one is convex.*

Lemma 3 $g(4) = 2$.

Proof. Let P_4 be a polygon of size 4, triangulated by adding a diagonal ab, as shown in Figure 5. By Observation 2, at least one of the vertices a or b is convex. Putting a guard g on the convex vertex is sufficient to guard P_4, as it covers both of the triangular faces. To ensure mutual visibility, we put another guard g' on a vertex that is visible from g. Therefore, $g(4) = 2$. □

Let X and Y be two interior disjoint triangles in the plane that share a side. We refer to the union of X and Y, which is a quadrilateral, by XY. Additionally, the shared side of X and Y is a diagonal in XY. In what follows, we refer to this as 'the diagonal' of XY.

Lemma 4 $g(5) = 2$.



(a) b is convex (b) b is reflex

Figure 6: Two cases for guarding P_5.

Proof. Let P_5 be a polygon of size 5. P_5 can have at most two reflex vertices, as the sum of the internal angles cannot exceed 540°. We triangulate P_5 by adding two non-intersecting diagonals and observe that its dual tree is a path of three nodes. Let B be the triangle corresponding to the middle node of the path, while triangles A and C correspond to its adjacent nodes, as shown in Figure 6. Let a, b, and c be the vertices of triangle B, such that b is shared between the two diagonals, a is shared between A and B, and c is shared between B and C. We have two cases:

- b is a convex vertex of P_5: We place a guard on b to cover the entire polygon, and another guard on another vertex to mutually see b, as in Figure 6(a).
- b is a reflex vertex of P_5: Then, one of a and c is convex. Without loss of generality, let c be the convex vertex. We place a guard on c. Let α be the portion of the angle of vertex b that lies in quadrilateral AB, as presented in Figure 6(b). By Observation 2, either α or the angle at vertex a is convex. We put one guard to cover this convex angle. This guard and c are mutually visible. □

A proof of the following lemma is given in the full version of the paper.

Lemma 5 *$g(6) = 3$.*

Lemma 6 *$g(7) = 4$.*

Proof. This statement is verified as there is a diagonal in every 7-gon that splits the polygon into a pentagon and a quadrilateral. By Lemmas 3 and 4, any quadrilateral and any pentagon can be guarded by 2 guards. Therefore, we divide the 7-gon into a pentagon and a quadrilateral, and we guard each with 2 guards separately, resulting in a total number of 4 guards. □

3 Orthogonal Polygons

In this section, we study the guarded guard problem for orthogonal polygons. In Section 3.1 we uncover some properties of orthogonal polygons. We will use these properties in our proof of the $\frac{n}{2}$ upper bound in Section 3.2. In the full version of the paper, we present our lower bound by exhibiting a family of polygons that require $\frac{3n-4}{7}$ guards for an arbitrary large n.

3.1 Preliminaries

We prove some properties of quadrangulated orthogonal polygons that will be used later in Section 3.2. However, these properties are of independent interest.

Observation 3 *Any simple polygon that admits a quadrangulation has an even number of vertices.*

Lemma 7 *Let P be an orthogonal polygon and Q be a quadrangulation of P. Any diagonal d of Q partitions P into two quadrangulated polygons P_1 and P_2 such that in each of P_1 and P_2 the edges that are adjacent to d are parallel to each other.*

Proof. Due to symmetry, we prove the statement only for P_1. Since P is orthogonal, its edges alternate between horizontal and vertical. Consequently, all edges of P_1 except d are orthogonal. The polygon P_1 admits a quadrangulation, which is inherited from Q. Thus, by observation 3, it has an even number of vertices and edges. Therefore, the two edges that are incident to d are either horizontal or vertical, and thus parallel to each other. □

The following observation is implied from Lemma 7.

Observation 4 *Let P be an orthogonal polygon and Q be a quadrangulation of P. Then every diagonal of Q is incident to at least one reflex vertex of P.*

Lemma 8 *Let $d = (a, b)$ be any diagonal in Q that divides P into two polygons P_1 and P_2. Let α_1, β_1 be the angles of P_1 at a and b, respectively, and let α_2, β_2 be the angles of P_2 at a and b, respectively. Then the following statements hold:*

1. $\min\{\alpha_1, \beta_1\} \leqslant \pi$ *and* $\min\{\alpha_2, \beta_2\} \leqslant \pi$
2. $\alpha_1 + \beta_1 = \pi$ *or* $\alpha_2 + \beta_2 = \pi$.

Proof. By Observation 4 one endpoint of d, say b, is a reflex vertex. We consider two cases:

1. *a is a convex vertex.* In this case the angle at a is $\frac{\pi}{2}$; see Figure 7(a). Statement 1 holds because α_1 and α_2 are both smaller than $\frac{\pi}{2}$. To verify statement 2, observe that the two edges of P_1 that are incident to d are on the same side of the line through d. This and the fact that these edges are parallel (by Lemma 7) imply that $\alpha_1+\beta_1 = \pi$. We get $\alpha_2+\beta_2 = \pi$ by a similar argument.

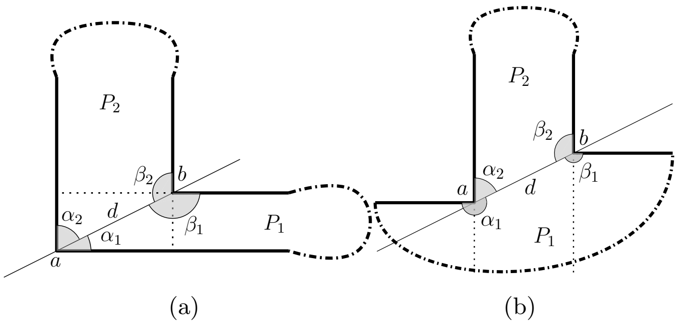


Figure 7: Possible configurations of a diagonal $d = (a, b)$ of P in Q.

2. *a is a reflex vertex.* In this case the two edges of, say P_2, that are incident to d are on the same side of the line through d, and the corresponding two edges in P_1 are on opposite sides of the line. See Figure 7(b). Therefore $\alpha_2 + \beta_2 = \pi$—this proves statement 2 and the second part of statement 1. It remains to show that $\min\{\alpha_1, \beta_1\} \leqslant \pi$. This statement is also true because $\alpha_1 + \beta_1 = 2\pi$ as the two edges of P_1, incident to d, lie on different sides of the line through d. □

3.2 The Upper Bound

In this section, we prove that any n-vertex orthogonal polygon P can be mutually guarded by at most $\frac{n}{2}$ guards. Hence, $\bar{g}(n) \leqslant \frac{n}{2}$. Note that n must be an even number, as P is orthogonal.

Theorem 9 *$\bar{g}(n) \leqslant \frac{n}{2}$, for every even $n \geqslant 4$.*

Proof. Let P be an orthogonal polygon, Q be a quadrangulation of P, and T be the dual tree of Q. Note that T has $\frac{n-2}{2}$ vertices of degree at most 4. We root T by taking an arbitrary leaf as the root. Thus, each node has at most 3 children (See Figure 8).

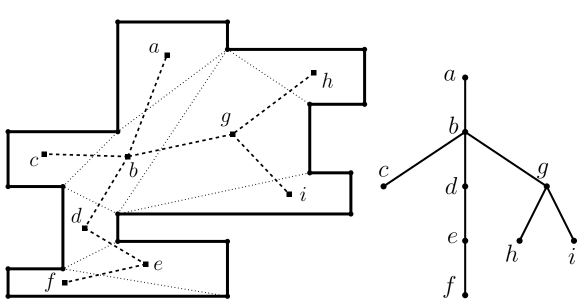


Figure 8: A quadrangulation and its dual tree.

Our proof is by partitioning T into connected subtrees with at least 2 and at most 4 nodes, and possibly one subtree of size 1. We partition T as follows:

Let l be a deepest leaf in T, i.e. a node with maximum distance from the root, and p be the parent of l. Let $T(p)$ be the subtree of T rooted at p, and note that $2 \leqslant |T(p)| \leqslant 4$ (assuming T has at least two nodes). We remove the vertices of $T(p)$ from T and repeat the above process. In the last iteration, the tree T may contain only one vertex which is the root, and we take it as a subtree with one node.

Consider any subtree $T(p)$ obtained after the above partitioning of T. Let polygon $\mathcal{P}$ be the union of quadrilaterals in Q corresponding to the nodes of $T(p)$. We guard $\mathcal{P}$ for each subtree separately. This would give a guarding of P. Assuming $2 \leqslant |T(p)| \leqslant 4$, we have $|\mathcal{P}| \in \{6, 8, 10\}$, as the number of vertices of $\mathcal{P}$ is given by $|\mathcal{P}| = 2|T(p)| + 2$.

In Lemmas 11, 12, and 13 we will show how to guard the corresponding polygon $\mathcal{P}$ of a subtree $T(p)$ with $|T(p)|$ guards. In case $T(p)$ is a 1-node tree, we guard $\mathcal{P}$ with 2 guards. This would give a total of $|T| + 1$ guards, which is $\frac{n-2}{2} + 1 = \frac{n}{2}$. □

Observe that if $|T(p)| = 1$, then $|\mathcal{P}| = 4$, in which case $\mathcal{P}$ can be guarded by 2 guards. In Lemmas 11, 12, and 13 we show how to guard a polygon $\mathcal{P}$ for a subtree $T(p)$, where $|T(p)| \in \{2, 3, 4\}$ and consequently $|\mathcal{P}| \in \{6, 8, 10\}$. In Figure 9, all possible configurations of $T(p)$ alongside a sample representation of its corresponding $\mathcal{P}$ are shown. Throughout the proofs, we use $\mathcal{P}$ to represent the union of the quadrilaterals that correspond to nodes of $T(p)$. We use $|\mathcal{P}|$ to denote the number of vertices of $\mathcal{P}$.

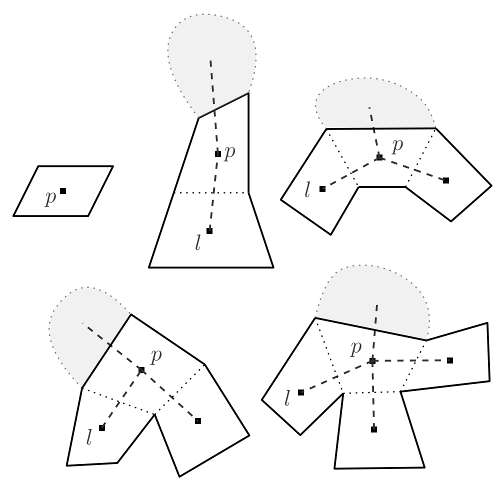


Figure 9: Possible polygonal patterns for $T(p)$.

Notice that the angles α_i and β_i in Lemma 8 are defined by the diagonals of Q and the boundary of P (not necessarily orthogonal). Also, notice that $\mathcal{P}$ is a simple polygon formed by some quadrilaterals in Q. Therefore, if we define α'_i and β'_i analogous to α_i and β_i but with respect to the diagonals of Q and the boundary of $\mathcal{P}$, then $\alpha'_i \leq \alpha_i$ and $\beta'_i \leq \beta_i$. Therefore, the following Corollary is valid.

Corollary 10 *Let $\mathcal{P}$ be a simple polygon formed by some quadrilaterals in Q. Let $d = (a, b)$ be any diagonal in Q that divides $\mathcal{P}$ into two polygons $\mathcal{P}_1$ and $\mathcal{P}_2$. Let α'_1, β'_1 be the angles of $\mathcal{P}_1$ at a and b, respectively, and let α'_2, β'_2 be the angles of $\mathcal{P}_2$ at a and b, respectively. Then the following statements hold:*

1. $\min\{\alpha'_1, \beta'_1\} \leqslant \pi$ *and* $\min\{\alpha'_2, \beta'_2\} \leqslant \pi$
2. $\alpha'_1 + \beta'_1 \leqslant \pi$ *or* $\alpha'_2 + \beta'_2 \leqslant \pi$.

While we only use the first statement in our proofs, we keep the second statement due to its independent interest and potential applications.

Lemma 11 *Two guards are sufficient for every $\mathcal{P}$ where $|\mathcal{P}| = 6$.*

Proof. A polygon $\mathcal{P}$ with 6 vertices consists of two quadrilaterals of Q, say A and B. Let a and b be the shared vertices between A and B—ab is the shared diagonal. We place one guard on a to cover A and one guard on b to cover B. Such a placement exists because A and B are convex quadrilaterals. The two guards are visible to each other as both cover the diagonal ab. □

Lemma 12 *Three guards are sufficient for every $\mathcal{P}$ where $|\mathcal{P}| = 8$.*

Proof. A polygon $\mathcal{P}$ with 8 vertices contains three quadrilaterals of Q. Let A be the quadrilateral corresponding to the child node l, B be the quadrilateral corresponding to the parent node p, and C be the quadrilateral corresponding to the sibling of l in $T(p)$. We consider two cases:

1. Quadrilaterals A and C share a vertex:

   Let b be the vertex shared between A and C, and let ab be the diagonal that separates A and B, and let bc be the diagonal that separates B and C, as in Figure 10(a). We place one guard on a to cover A, and one guard on c to cover C. As the quadrilaterals are convex, both guards cover vertex b. We place a third guard on b to cover B. Thus, the entire $\mathcal{P}$ is guarded, and the guards on a and c are mutually visible from the guard on b.

2. Quadrilaterals A and C has no common vertex:

   Let ab_1 be the diagonal that separates A and B and let cb_2 be the diagonal that separates B and C, such that ac corresponds to the edge of T that connects p to its parent, as in Figure 10(b). By statement 1 of Corollary 10, the angle at one of the endpoints of ac in polygon $\mathcal{P}$ is at most π. Due to symmetry, let a be this endpoint. By placing a guard on a we cover both A and B. We place one guard on b_2 to cover C. We place a third guard on c to cover B. The guards on a and c see each other through ac, and the guards on c and b_2 see each other through cb_2. Hence, $\mathcal{P}$ is mutually covered by three guards.

□

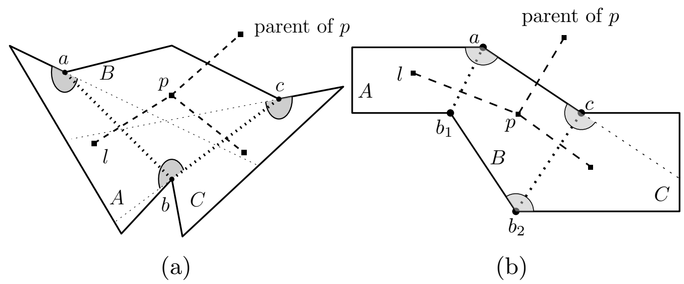


Figure 10: A and C share (a) one vertex (b) no vertex.

A proof of the following lemma is given in the full version of the paper.

Lemma 13 *Four guards are sufficient for every $\mathcal{P}$ where $|\mathcal{P}| = 10$.*

4 Open Problems

One natural problem is to improve any of the bounds given for $g(n)$ and $\bar{g}(n)$. We believe the true bounds are closer to our lower bounds.

Another research direction is to relax any of the constraints that is imposed on a feasible guard set in Section 1.2. For example one may allow outward guards, because in some cases they lead to smaller number of guards, as in Figure 11. Alternatively one may allow the two guards that are placed at the same vertex to be mutually visible to each other.

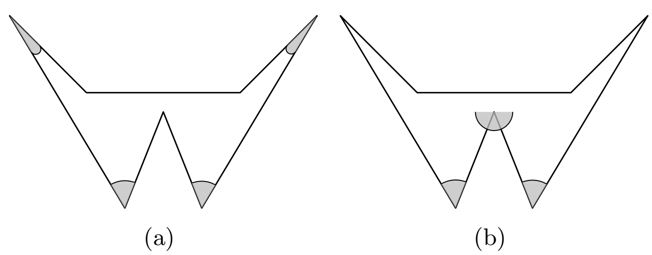


Figure 11: Outward guarding can result in less guards

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