The Number of Non-overlapping Unfoldings in Convex Polyhedra*

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Abstract

An unfolding of a polyhedron is a flat polygon obtained by selecting candidate cutting lines, cutting along them, and flattening the faces onto the plane. Several methods have been proposed to efficiently count the number of unfoldings by treating the candidate cutting lines as a graph. One such method is to use a zero-suppressed binary decision diagram (ZDD), a compact data structure for representing families of sets. However, some unfoldings overlap depending on the shape of the polyhedra and how they are unfolded. In such cases, two distinct faces intersect or touch in the plane, making it impossible to embed the unfolding in the plane. In this study, we address the problem of counting the number of non-overlapping edge unfoldings in convex polyhedra. We propose a ZDD-based algorithm that excludes overlapping unfoldings by removing their minimal overlapping patterns. Our method applies to both edge and lattice unfoldings, and we present experimental results on several convex polyhedra.

1 Introduction

An unfolding of a polyhedron is a flat polygon obtained by selecting candidate cutting lines, cutting along them, and flattening the faces onto the plane. Depending on the shape of the polyhedron and how it is unfolded, the resulting unfolding may have overlaps, i.e., two distinct faces overlap, or their boundaries are in touch (see Figure 1). When the candidate cutting lines are restricted to the edges of the polyhedron, the unfolding is called an *edge unfolding.* Shephard proposed the following conjecture about edge unfoldings.

Conjecture 1 ([13]) *For any convex polyhedron, at least one non-overlapping edge unfolding exists.*

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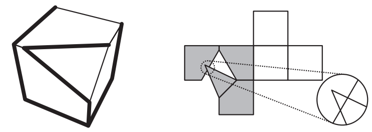


Figure 1: A cube with truncated corners and its overlapping unfolding [11].

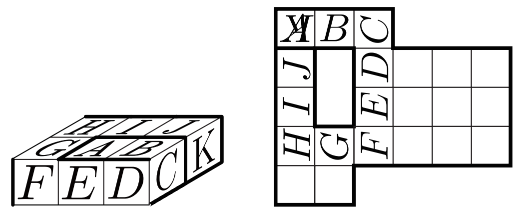


Figure 2: Examples of overlapping lattice unfoldings for $(1, 2, 3)$-cuboid. Faces A and K overlap.

This conjecture is still unsolved, and some studies to solve it are ongoing. One such study is determining whether an overlapping edge unfolding exists for a given polyhedron. This line of investigation is based on the idea that one possible approach to a negative resolution of Conjecture 1 is to find a convex polyhedron for which every edge unfolding overlaps. Shiota and Saitoh presented an algorithm called "rotational unfolding" that can quickly find an overlapping edge unfolding of a polyhedron [15]. This algorithm can determine whether the edge unfolding of a polyhedron overlaps. For convex regular-faced polyhedra (convex polyhedra in which every face is a regular polygon), the existence of overlapping edge unfoldings has been completely demonstrated [2, 4, 6, 7, 15].

General unfoldings, which allow cuts across the faces and edges of a polyhedron, have been studied. Some general unfoldings permit cuts only along specific candidate lines drawn on the faces. One such example is the lattice unfolding of a cuboid formed by connecting multiple $(1, 1, 1)$-cubes [10]. In lattice unfolding, we cut along the edges of the lattice formed by unit squares. For lattice unfoldings of cuboids, the existence of overlapping unfoldings has also been fully demonstrated [10, 14] (see Figure 2 for an example).

The problem of counting the number of unfoldings has been studied in previous works. Schevon experimentally showed that, for randomly generated convex polyhedra,

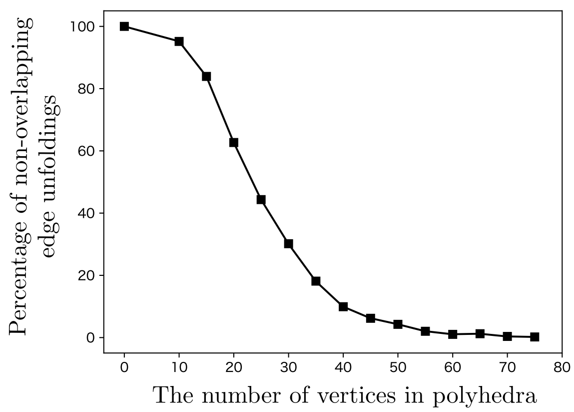


Figure 3: Shevon's experiment on randomly generated convex polyhedra. Each point shows the average percentage of non-overlapping edge unfoldings, computed from 1,000 random unfoldings for each of 5 polyhedra.

the percentage of non-overlapping edge unfoldings decreases as the number of vertices increases [12] (see Figure 3).

The number of unfoldings (including those with overlaps) is equal to the number of *cutting trees* (trees satisfying specific conditions on the candidate cutting lines) [3, 10]. Horiyama et al. counted the number of cutting trees using *zero-suppressed binary decision diagrams (ZDDs)*, a compact data structure for representing families of sets [5, 7].

Our contributions. Herein, we propose an algorithm for counting the number of non-overlapping unfoldings of a given polyhedron using ZDDs and operations over them. The algorithm first enumerates the *minimal overlapping partial unfoldings (MOPUs)*, which are minimal units of unfoldings obtained through the rotational unfolding [15] (the gray faces in Figure 1 correspond to this). Then, we construct a ZDD representing non-overlapping unfoldings by removing the unfoldings containing MOPUs from the ZDD of all possible unfoldings. In this paper, we apply the proposed algorithm to edge unfoldings of convex regular-faced polyhedra and lattice unfoldings of cuboids, and we present the number and percentage of non-overlapping unfoldings for each convex polyhedron (see Tables 1 and 2 for selected results). These results suggest that the number of non-overlapping unfoldings is more significantly affected by the number of faces comprising each MOPU than by the number of MOPUs themselves.

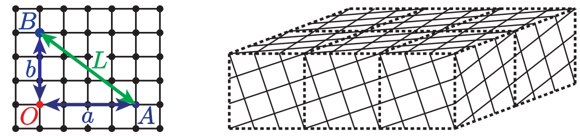
2 Preliminaries

2.1 Edge unfolding of polyhedra

Let Q be a polyhedron. An *unfolding* of the polyhedron Q is a flat polygon formed by cutting Q's edges or faces and unfolding it into a plane. An *edge unfolding* of Q is an unfolding formed by cutting only edges. Q can be viewed as a graph $G_Q = (V_Q, E_Q)$, where V_Q is a set of vertices and E_Q is a set of edges. We have the following lemma for an edge unfolding of Q.

Lemma 1 ([3] Lemma 22.1.1) *The cutting lines of an edge unfolding for Q form a spanning tree of G_Q.*

We say that two distinct polygons *overlap* if there exists a point p contained in both of the polygons. Note that any point on a boundary is included in the polygons in this paper. An unfolding is *overlapping* if there exists a pair of distinct faces such that the faces overlap. *Rotational unfolding* is an efficient algorithm for determining whether a given polyhedron Q has an overlapping edge unfolding [15].



(a) (b)

Figure 4: (a) Definition of the edge length L of a cube. (b) An $(\sqrt{10}, 2\sqrt{10}, 3\sqrt{10})$-cuboid.

2.2 Lattice unfolding of cuboids

Let us consider a square lattice where each square has an area of 1×1. Let $A = (a, 0)$ and $B = (0, b)$ be lattice points, where $a \in \mathbb{N}^+$, $b \in \mathbb{N}$, and $a \geq b$, as illustrated in Figure 4 (a). We define $L = \sqrt{a^2 + b^2}$ as the length of the segment AB. An (xL, yL, zL)*-cuboid* is then defined as a box whose edge lengths are xL, yL, and zL along the x-, y-, and z-axes, respectively, where $x, y, z \in \mathbb{Z}^+$. Figure 4 (b) shows an example of a lattice cuboid. Here, a square enclosed by a solid line corresponds to one unit of the original square grid, and a square enclosed by a dotted line has a side length of L.

A *lattice unfolding* of a cuboid C is a planar shape obtained by cutting along the edges of unit squares on the faces of the cuboid. C can be viewed as a graph $G_C = (V_C, E_C)$, where V_C is the set of lattice points (i.e., integer-coordinate points) on the surface of C, and E_C is the set of edges between them. We have the following lemma for a lattice unfolding of C.

Lemma 2 (Figure 5, and [10] Theorems 1 and 3) *Let $S(V_C) \subseteq V_C$ be the set of lattice points located at the vertices of C. Then, the following are equivalent for a subgraph $G_L \subseteq G_C$:*

(1) *A lattice unfolding can be obtained by cutting along G_L.*

(2) *G_L is a tree that satisfies $S(V_C) \subseteq G_L$, and for any vertex v in G_L, if the degree of vertex v is 1, then $v \in S(V_C)$.*

Table 1: The number and percentage of non-overlapping edge unfoldings for convex regular-faced polyhedra (excerpt).

Archimedean solids	$\|V\|$	$\|E\|$	$\|F\|$	#(MOPUs)	#(Edge unfoldings)	#(Non-overlapping edge unfoldings)	Pct.(%)
Truncated dodecahedron	60	90	32	120	4,982,259,375,000,000,000	1,173,681,002,295,455,040	23.56
Truncated icosahedron	60	90	32	240	375,291,866,372,898,816,000	371,723,160,733,469,233,260	99.05
Archimedean n-prisms	$\|V\|$	$\|E\|$	$\|F\|$	#(MOPUs)	#(Edge unfoldings)	#(Non-overlapping edge unfoldings)	Pct.(%)
27-prism	54	81	29	216	37,403,957,244,654,675	35,348,297,730,550,335	94.50
28-prism	56	84	30	336	144,763,597,316,784,768	136,369,030,045,792,768	94.20
29-prism	58	87	31	580	559,560,282,425,278,229	377,763,966,359,384,333	67.51
30-prism	60	90	32	720	2,160,318,004,043,512,500	1,457,228,998,699,944,660	67.45
Archimedean m-antiprisms	$\|V\|$	$\|E\|$	$\|F\|$	#(MOPUs)	#(Edge unfoldings)	#(Non-overlapping edge unfoldings)	Pct.(%)
16-antiprism	32	64	34	64	151,840,963,183,392	146,378,600,602,880	96.40
17-antiprism	34	68	36	204	1,105,779,284,582,146	989,008,190,008,480	89.44
18-antiprism	36	72	38	432	8,024,954,790,380,544	1,517,682,139,108,200	18.91
19-antiprism	38	76	40	456	58,059,628,319,357,318	10,550,126,657,845,736	18.17

Table 2: The number and percentage of non-overlapping lattice unfoldings for cuboids (excerpt).

					Faces-in-touch			Edges-in-touch			Vertices-in-touch		
Cuboids	$\|V\|$	$\|E\|$	$\|F\|$	# (Lattice unfoldings)	#(MOPUs)	# (No(faces) unfoldings)	Pct.(%)	#(MOPUs)	# (No(edges) unfoldings)	Pct.(%)	#(MOPUs)	# (No(vertices) unfoldings)	Pct.(%)
$(1,1,1)$	8	12	6	384	0	384	100.00	0	384	100.00	0	384	100.00
$(1,1,2)$	12	20	10	12,124	0	12,124	100.00	0	12,124	100.00	32	11,484	94.72
$(1,1,3)$	16	28	14	240,304	16	240,240	99.97	80	238,432	99.22	304	212,920	88.60
$(1,1,4)$	20	36	18	3,708,380	80	3,705,820	99.93	512	3,644,600	98.28	1,232	3,075,400	82.93
$(1,1,5)$	24	44	22	49,206,176	208	49,156,592	99.90	1,504	47,970,720	97.49	3,408	38,043,936	77.32
$(1,1,6)$	28	52	26	592,188,796	464	591,487,340	99.88	3,808	573,122,568	96.78	8,448	424,509,028	71.68
$(1,1,7)$	32	60	30	6,671,469,328	1,104	6,663,017,440	99.87	9,360	6,409,933,496	96.08	20,432	4,407,661,888	66.07
$(1,1,8)$	36	68	34	71,772,242,780	2,704	71,679,140,716	99.87	22,912	68,429,543,676	95.34	49,456	43,445,829,708	60.53
$(1,1,9)$	40	76	38	747,116,459,968	6,544	746,143,953,328	99.87	55,584	706,395,487,984	94.55	119,504	412,096,369,696	55.16
$(1,1,10)$	44	74	42	7,593,452,118,844	15,760	7,583,621,450,924	99.87	134,368	7,114,772,651,372	93.70	288,416	3,797,487,539,408	50.01
$(1,2,2)$	18	32	16	1,675,184	0	1,675,184	100.00	32	1,553,536	92.74	128	1,228,824	73.35
$(1,2,3)$	24	44	22	131,478,632	544	130,212,292	99.04	1,648	111,177,796	84.56	3,312	75,653,292	57.54
$(1,2,4)$	30	56	28	7,692,072,382	14,920	7,528,985,598	97.88	32,048	5,970,306,978	77.62	52,960	3,535,269,930	45.96
$(1,2,5)$	36	68	34	375,631,947,892	141,816	364,028,460,124	96.91	291,736	270,654,176,916	72.05	449,552	140,837,624,986	37.49
$(2,2,3)$	34	64	32	203,758,066,112	5,824	196,470,177,268	96.42	19,392	109,840,848,592	53.91	34,704	48,990,450,676	24.04
$(\sqrt{2},2\sqrt{2},2\sqrt{2})$	34	64	32	207,761,826,744	13,296	198,307,283,288	95.45	45,776	135,619,116,108	65.28	76,432	67,737,527,156	32.60

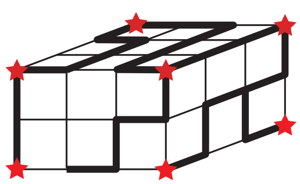
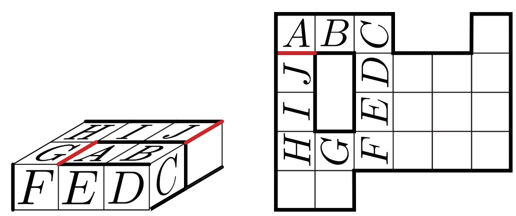
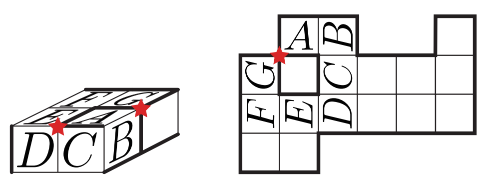


Figure 5: An example of a cutting line in a $(2,3,3)$-cuboid. The cutting line forms a tree that includes all eight lattice cuboid vertices (the starred ones).



(a) Edges-in-touch unfolding in the $(1,2,3)$-cuboid



(b) Vertices-in-touch unfolding in the $(1,2,2)$-cuboid

Figure 6: Overlapping lattice unfolding [10,14]. In the cuboids shown in (a) and (b), the red edges and the starred vertices, respectively, are not in touch.

In a lattice unfolding, the original cuboid's unit squares are arranged on a plane, with their edges connected. The relationship between any pair of unit squares that are not adjacent on the original cuboid is classified as follows [14]:

(1) Overlap in the same position (Figure 2).

(2) Share one edge (Figure 6 (a)).

(3) Share one vertex (Figure 6 (b)).

(4) Do not share any edges or vertices.

Herein, we say that a lattice unfolding is *faces-in-touch*, *edges-in-touch*, or *vertices-in-touch* if it has a pair of unit squares satisfying condition (1), (2), or (3), respectively. If any of the conditions (1)-(3) is satisfied, we say the *lattice unfolding is overlapping*. Otherwise, when all pairs of unit squares that are not adjacent on the original cuboid satisfy condition (4), we say the *lattice unfolding is non-overlapping*. Note that for any cuboid,

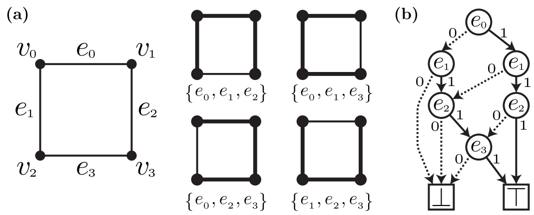


Figure 7: (a) The graph C_4 and its spanning trees. (b) A ZDD representing the spanning trees of C_4.

the following strict inclusion relations hold among the families of unfoldings: $\mathcal{U}_f \subset \mathcal{U}_e \subset \mathcal{U}_v$, where $\mathcal{U}_f$, $\mathcal{U}_e$, and $\mathcal{U}_v$ denote the sets of unfoldings with face-in-touch, edge-in-touch, and vertex-in-touch, respectively.

2.3 Counting the number of unfoldings

From Lemmas 1 and 2, the number of unfoldings (including those with overlaps) is equal to the number of trees satisfying specific conditions on the candidate cutting lines (hereafter called *cutting trees*). One method for counting cutting trees is using a *Zero-suppressed Decision Diagram (ZDD)* [5]. A ZDD is a data structure that compactly represents a family of sets as a directed acyclic graph [9] (see the example in Figure 7). It consists of two types of nodes: *terminal nodes* with out-degree zero ($\top$, $\bot$), and *branching nodes* labeled by elements of the set. Each branching node has two outgoing edges: a 1-*edge*, which indicates the inclusion of the labeled element, and a 0-*edge*, which indicates its exclusion. A ZDD has a unique *root node* with no incoming edges, and each path from the root to $\top$ corresponds to a specific set.

3 Counting algorithm for the number of non-overlapping unfoldings

In this section, we present an algorithm for counting non-overlapping unfoldings. Section 3.1 introduces the notion of *minimal overlapping partial unfoldings (MOPU)*, which serve as the basis for our algorithm. In Section 3.2, we present a ZDD-based algorithm for counting only non-overlapping unfoldings.

3.1 Minimal overlapping partial unfoldings

We begin by introducing several notions used to define minimal overlapping partial unfoldings. Let Q be a polyhedron. Two faces in Q are *adjacent* if they are connected through a common cutting line. The *dual graph* of Q is a graph $G_D = (V_D, E_D)$, where each vertex in V_D corresponds to a face of Q, and two vertices are connected by an edge in E_D if and only if the corresponding faces are adjacent. A *partial unfolding* is a flat polygon consisting of a set of faces that correspond to a connected induced subgraph of G_D. A *minimal partial unfolding* is a partial unfolding consisting of the faces along a simple path between two vertices in G_D. A *minimal overlapping partial unfolding (MOPU)* is a minimal partial unfolding in which the two end faces overlap. Figure 8 shows examples of MOPUs in edge unfoldings. In each case, the two end faces overlap, and removing any face would break the connectivity of the unfolding, which makes them minimal. The partial unfoldings consisting of the lettered faces in Figures 2 and 6 are also examples of MOPUs in lattice unfoldings. MOPUs in convex regular-faced polyhedra and lattice cuboids can be enumerated using the rotational unfolding [14, 15].

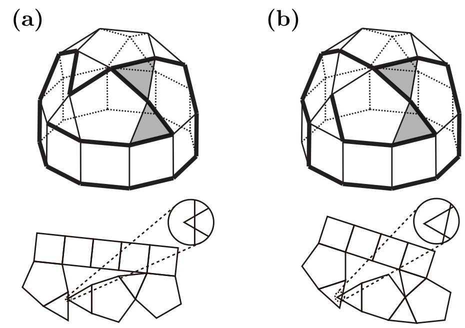


Figure 8: Examples of MOPUs in a convex regular-faced polyhedron (J21). The gray faces indicate the end faces.

3.2 ZDD-based algorithm

In this section, we describe an algorithm for counting the number of non-overlapping unfoldings of a given polyhedron. As described in Section 2.3, the number of cutting trees can be counted by constructing a ZDD $\mathcal{Z}_{\mathcal{T}}$. However, some cutting trees result in overlapping unfoldings, depending on the shape of the polyhedron. In such cases, $\mathcal{Z}_{\mathcal{T}}$ includes these overlapping unfoldings. To efficiently remove overlapping unfoldings from $\mathcal{Z}_{\mathcal{T}}$, we use *subsetting*, a ZDD operation [8]. The subsetting technique constructs a new ZDD $\mathcal{Z}_{\mathcal{N}}$ by extracting the family of sets that satisfy the constraint $\mathcal{C}$ from ZDD $\mathcal{Z}$.

We now present a method for removing overlapping unfoldings by subsetting. Let U be a partial unfolding, and let $NC[U]$ be the set of edges that are not cut when unfolding the polyhedron. We denote by k the number of MOPUs, enumerated by the rotational unfolding. The following lemma holds for any MOPU M_i ($1 \le i \le k$).

Lemma 3 *If an unfolding U satisfies $NC[M_i] \subseteq NC[U]$, then U is an overlapping unfolding.*

Proof. Let the sequence of faces in MOPU M_i be $\langle f_1, f_2, \ldots, f_\ell \rangle$, and let e_j be the edge shared between each pair of adjacent faces f_j and f_{j+1} (where the faces

f_1 and f_ℓ overlap). Since $NC[M_i]$ represents the set of uncut edges in the partial unfolding M_i, we can write $NC[M_i] = \{e_1, e_2, \ldots, e_{\ell-1}\}$. On the other hand, from the condition $NC[M_i] \subseteq NC[U]$, it follows that the set $\{e_1, e_2, \ldots, e_{\ell-1}\}$ must be included in the unfolding U. Therefore, the sequence of faces $\langle f_1, f_2, \ldots, f_\ell \rangle$ appears in U, indicating that U has overlaps. □

From Lemma 3, removing the family of sets $\mathcal{U}_i$ (which represents unfoldings containing the MOPU M_i) from the ZDD $\mathcal{Z}_\mathcal{T}$ yields a ZDD that represents only unfoldings that do not include the structure of M_i. On the other hand, to construct the family of sets $\mathcal{U}_i$ representing unfoldings that include MOPU M_i, we need a ZDD representing the family of cutting trees that contain $NC[M_i]$. However, by applying the following lemma, we can avoid constructing a ZDD representing all cutting trees containing $NC[M_i]$, and instead create a simpler ZDD.

Lemma 4 *Given the family of sets $\mathcal{Z}_\mathcal{T}$ representing all unfoldings, the following conditions are equivalent:*

(1) *The family of sets obtained by removing the unfoldings that include MOPU M_i from $\mathcal{Z}_\mathcal{T}$.*

(2) *The family of sets obtained by removing the family $\mathcal{F}_i = \{NC[M_i] \cup E' \mid E' \subseteq E \setminus NC[M_i]\}$, which contains all subsets that include $NC[M_i]$, from $\mathcal{Z}_\mathcal{T}$.*

Proof. From the condition, we know that $\mathcal{U}_i \subseteq \mathcal{F}_i$. Now, if we define $\mathcal{N}_i = \mathcal{F}_i \setminus \mathcal{U}_i$, then $\mathcal{N}_i$ contains no sets that represent unfoldings, meaning $\mathcal{N}_i \not\subseteq \mathcal{Z}_\mathcal{T}$. Therefore, we have the following equivalence:

$$\mathcal{Z}_\mathcal{T} \setminus \mathcal{F}_i = \mathcal{Z}_\mathcal{T} \setminus (\mathcal{N}_i \cup \mathcal{U}_i) = \mathcal{Z}_\mathcal{T} \setminus \mathcal{U}_i,$$

which completes the proof. □

Note that the removal procedure based on Lemma 4 is order-independent: if an unfolding contains the uncut edge set of any MOPU M_i, it will be excluded at that step, regardless of whether it also contains other $NC[M_j]$. Therefore, the resulting ZDD correctly retains only non-overlapping unfoldings.

Therefore, we can construct a ZDD that represents non-overlapping unfoldings by following three steps:

Step 1. Enumerate MOPUs using rotational unfolding [14, 15].

Step 2. Construct the ZDD $\mathcal{Z}_\mathcal{T}$ that represents all possible unfoldings, and for each i $(1 \le i \le k)$, construct a ZDD $\mathcal{F}_i$ representing the family of all sets that do not simultaneously contain all elements of $NC[M_i]$.

Step 3. Apply the subsetting technique [8] on $\mathcal{Z}_\mathcal{T}$ using the constraints from each $\mathcal{F}_i$, to construct a ZDD $\mathcal{Z}_\mathcal{N}$ that excludes MOPUs M_1 through M_k.

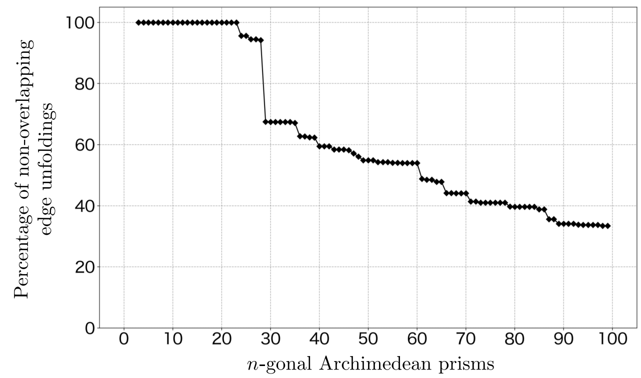


Figure 9: The percentage of non-overlapping edge unfoldings in Archimedean prisms.

4 Computational experiments

Here, we present the results of applying the algorithm for counting non-overlapping unfoldings to the edge unfoldings of 175 convex regular-faced polyhedra and the lattice unfoldings of 23 cuboids. We used the TdZdd library[1] to construct the ZDD $\mathcal{Z}_\mathcal{T}$, which represents all unfoldings, the ZDD $\mathcal{F}_i$, which represents the family of sets containing all elements of $NC[M_i]$, and applied the subsetting method. To enumerate MOPUs for the convex regular-faced polyhedra and lattice cuboids, we used rotational unfolding[2] [14, 15]. All computational experiments were conducted on an Intel(R) Xeon(R) CPU E5-2643 v4 at 3.40 GHz with 512 GB of memory, running CentOS 7.9[3].

4.1 The number of non-overlapping edge unfoldings for convex regular-faced polyhedra

As introduced in Section 1, Table 1 shows the number and percentage of non-overlapping edge unfoldings for selected convex regular-faced polyhedra[4]. Figures 9 and 10 show line plots of the percentage of non-overlapping edge unfoldings for Archimedean prisms and antiprisms, respectively. In each figure, the horizontal axis indicates the number of sides n (or m) of the base polygon, and the vertical axis shows the percentage of non-overlapping edge unfoldings.

From the results of these experiments, we can find the following. Let us compare the truncated icosahedron and the truncated dodecahedron in Archimedean solids (see Table 1, under Archimedean solids). Both polyhedra have the same number of vertices, edges, and faces,

[1] https://github.com/kunisura/TdZdd

[2] https://shiotatakumi.github.io/MyPage/library/RotationalUnfolding.html

[3] The source code and details on how it was tested can be found at https://shiotatakumi.github.io/MyPage/library/CountingNonoverlappingUnfoldings.html.

[4] The full version can be found in https://shiotatakumi.github.io/MyPage/contents/250813-CCCG-2025.html#A

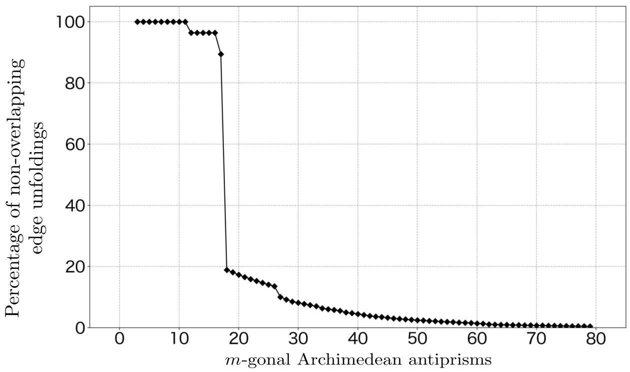


Figure 10: The percentage of non-overlapping edge unfoldings in Archimedean antiprisms.

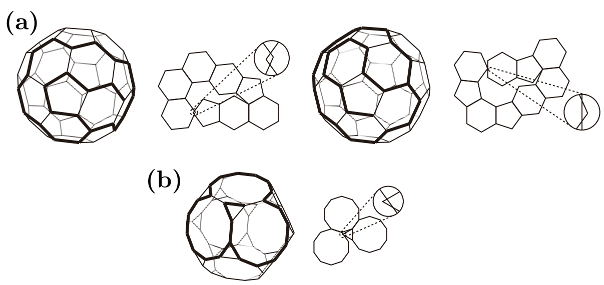


Figure 11: MOPUs in (a) Truncated icosahedron and (b) Truncated dodecahedron, excluding rotational and mirror equivalents [7, 15].

but the truncated icosahedron has more MOPUs than the truncated dodecahedron. However, the truncated dodecahedron has a lower percentage of non-overlapping edge unfoldings. The MOPUs of the truncated icosahedron consist of eight or nine faces (Figure 11 (a)), whereas the truncated dodecahedron has a MOPU composed of only four faces (Figure 11 (b)). Thus, we can observe that the presence of MOPUs composed of fewer faces has a greater influence on reducing the percentage of non-overlapping edge unfoldings than the number of MOPUs.

This trend is also observed in Archimedean n-gonal prisms and m-gonal antiprisms. In both cases, the percentage of non-overlapping edge unfoldings significantly decreases at $n = 29$ and $m = 18$, respectively, coinciding with the appearance of MOPUs composed of fewer faces (see Figure 9 and 10; see also Table 1, under Archimedean prisms and antiprisms). Figures 12 and 13 show how the structure of MOPUs changes around the point where the percentage decreases.

4.2 The number of non-overlapping lattice unfoldings for cuboids

Table 2, which also appears in Section 1, shows the number and percentage of non-overlapping lattice unfoldings in selected (xL, yL, zL)-cuboids that do not have faces-in-touch, edges-in-touch, or vertices-in-touch, respectively[5]. In what follows, we use "No(xx)" to denote a lattice unfolding that does not have any xx-in-touch.

From the results for $(1, 1, z)$-cuboids $(1 \le z \le 10)$, we observe that the percentage of non-overlapping lattice unfoldings tends to decrease as z increases. Similar trends are also observed in other results.

Among lattice cuboids, there exist cuboids that have the same surface area but different volumes. Tables 3 and Table 4 show the percentages of lattice unfoldings without face-, edge-, or vertex-in-touch for cuboids with a surface area of 32 and 34, respectively. These tables indicate that the percentages of No(edges) and No(vertices) unfoldings tend to decrease as the volume increases. However, despite the increasing volume, the percentage of No(faces) unfoldings decreases by approximately 1% at surface area 32, but increases by about 3% at 34. Thus, while a larger volume tends to lower the percentage of non-overlapping lattice unfoldings, volume alone does not fully explain the trend[6].

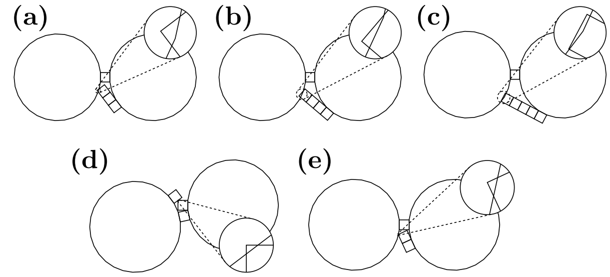


Figure 12: MOPUs in Archimedean prisms: (a)-(c) for 28-gonal and (a)-(e) for 29-gonal prisms, excluding rotational and mirror equivalents [15]. (a)-(c) consist of 6, 7, and 8 faces, respectively, and (d) and (e) each consist of 5 faces.

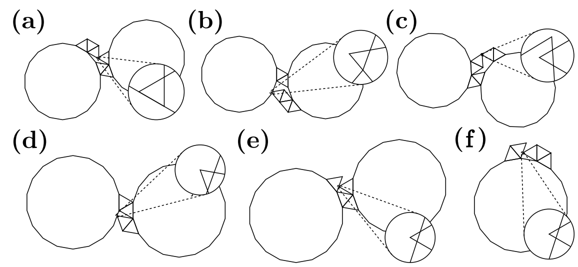


Figure 13: MOPUs in Archimedean antiprisms: (a)-(c) for 17-gonal and (a)-(f) for 18-gonal antiprisms, excluding rotational and mirror equivalents [15]. (a)-(c) each consist of 8 faces, and (d)-(f) each consist of 6 faces.

[5] The full version can be found in https://shiotatakumi.github.io/MyPage/contents/250813-CCCG-2025.html#B

[6] Similar trends are also observed for cuboids with other surface areas; see https://shiotatakumi.github.io/MyPage/contents/250813-CCCG-2025.html#C

Table 3: The percentage of non-overlapping lattice unfoldings for a cuboid with a surface area of 32.

Cuboids	Volume	No(faces)	No(edges)	No(vertices)
$(\sqrt{2}, 2\sqrt{2}, 2\sqrt{2})$	$8\sqrt{2}$	95.45	65.28	32.60
$(2, 2, 3)$	12	96.42	53.91	24.04

Table 4: The percentage of non-overlapping lattice unfoldings for a cuboid with a surface area of 34.

Cuboids	Volume	No(faces)	No(edges)	No(vertices)
$(1, 1, 8)$	8	99.87	95.34	60.53
$(1, 2, 5)$	10	96.91	72.05	37.49

5 Conclusion

In this paper, we proposed an algorithm to count the number of non-overlapping unfoldings using ZDDs and MOPU-based operations. We applied the algorithm to two types of unfoldings and obtained insights into the structural conditions under which the number of non-overlapping unfoldings decreases.

There are four main directions for future work. The first is to investigate the underlying reason for the observation in Section 4.2, where the percentage of No(faces) unfoldings does not decrease monotonically with increasing volume for cuboids with the same surface area. The second is to apply the proposed method to other classes of general unfoldings, such as pseudo-edge unfoldings [1]. The third is to investigate whether the proposed method can be extended to enumerate non-overlapping unfoldings of non-convex polyhedra. Since the method treats a polyhedron as a graph and does not depend on its geometric shape, it is expected to naturally generalize to non-convex cases. In this type of unfolding, the vertices correspond to the original vertices of the polyhedron, the edges are shortest geodesic paths between vertex pairs, and the surface is unfolded by cutting along these paths. The fourth is to pursue a theoretical analysis of the computational complexity of the proposed method. However, since the following problem remains open, providing a formal runtime bound is difficult:

Open problem 5 *Is it computationally hard to decide whether a given unfolding has overlaps?*

Addressing this problem would naturally precede any formal analysis of the method. These directions may provide new insights toward resolving Conjecture 1.

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